



Precise Determination of the Disturbing Potential Using Alternative Boundary Values

Rockville, Md.
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CONTENTS

Abstract	1
1. Introduction	1
2. Basic considerations	2
3. Truncation error	6
4. Alternative gravity field representations	15
5. A new type of geodetic boundary value problem	17
6. Relative comparison of space and terrestrial data	19
7. Remarks on combining space and terrestrial data	20
8. The basic zero-reference of height systems	21
9. Some numerical aspects	26
10. Numerical investigations	31
11. Discussion	40
12. Conclusion	43
13. Acknowledgment	43
Appendix A. Derivations of Hotine's $H(r,\psi)$ in its generalized form	44
Appendix B. Discussion of density model	49
Appendix C. A remark about secular Love numbers	64
References	68

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PRECISE DETERMINATION OF THE DISTURBING POTENTIAL USING ALTERNATIVE BOUNDARY VALUES

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ABSTRACT. Advanced gravimetric techniques can be applied for the precise determination of geodetic parameters. Data from the Global Positioning System (GPS) and from GRAVSAT-type satellites lend themselves to combinations and comparisons with terrestrial data. Such comparisons are of special interest for well-known systematic distortions in terrestrial data. In addition, new types of data lead to new types of boundary value problems and imply fundamental changes in geodetic concepts.

1. INTRODUCTION

Physical geodesy was dominated for decades by the search for explicit solutions to the free boundary value problem, where the unknown surface S of the Earth is determined together with the gravity potential in the space exterior to S . What attracted the most interest in these attempts was the various ways in which the free boundary value problem was replaced by a fixed boundary value problem. Molodensky's use of the telluroid as an approximate surface of the Earth to which the boundary values could be referred is the type of solution that received the greatest attention during the last two decades. (For details see Molodenskii et al. (1962).) Somewhat different definitions for the approximate surface were introduced by Krarup, Marussi, Grafarend, and others. More recently, the approach with the greatest theoretical interest is Sanso's solution. Moritz (1980) presents an excellent review of these methods. He also gives a good description of least-squares collocation solutions which can be applied to the problem.

On the other hand, many of the prerequisites usually applied to those solutions are no longer valid. In the future, astronomical coordinates will not generally be considered as observable quantities because classical astrogeodetic techniques will have become obsolete. We will never know surface gravity everywhere at the Earth's surface with the

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accuracy that is necessary in modern operational geodesy. In spite of the fact that even recent satellite tracking coordinates may be affected by systematic errors (in the past they have been affected by as much as several meters and more, implying uncertainties of the order of 10 mgal), we can expect future techniques to yield errors of less than 1 decimeter in coordinates, and mean gravity values of a 1°-by-1° block to about an accuracy of plus-or-minus a few milligals (Douglas et al. 1980). If the sea surface is surveyed by satellite altimetry, the deviation of the sea surface from the geoid (or equivalent reference surface) can be determined. Thus, the determination of the ocean geoid can be handled by itself without regard to the remainder of the Earth's surface. Therefore, the original geodetic boundary value problem has totally changed.

As soon as the disturbing potential T can be inferred for small blocks, such as 1° surface compartments, the derivative boundary value problem (Moritz 1980) loses its dominant role in physical geodesy. By using Runge's theorem, as pointed out by Krarup in various papers, we can analytically continue the potential T down to a Bjerhammar-type sphere in the Earth's interior and determine whatever derivative (or linear combination thereof) of T we need at any point exterior to that sphere.

However, various detailed problems have to be solved before we can consider the problems of physical geodesy in these new formulations. It is the aim of the following discussion to bridge the present and the future by outlining alternative methods by which optimal solutions can be achieved with both current and future data. It is not the aim of this study to investigate the accuracy with which the gravitational potential W is determined at the Earth's surface. Instead, this research seeks to determine the accuracy of the disturbing potential T based on studies such as those of Douglas et al. (1980), together with accuracy estimates of surface gravity, thus leading to accuracy estimates of combination solutions.

The downward continuation is not treated in detail, since, with the GRAVSAT data now available (which are supposed to yield gravity on a sphere in space or, at least, an almost evenly distributed set of gravity data close to a sphere in space), we encounter relatively simple downward continuation problems in geodetic applications.

2. BASIC CONSIDERATIONS

In Groten (1979, 1980) the Neumann boundary value problem was considered for the geoid or for a reference sphere, such as the Bjerhammar sphere. Using Hotine's formula (1969, p. 317, para. 33)

$$T(P) = \frac{R}{4\pi} \iint_S \delta g \, H ds \quad (1)$$

where

T = disturbing potential,

R = mean radius of the Earth,

P = a point on s ,

H = Neumann kernel function corresponding to Hotine's function
eq. (29.17) (Hotine 1969, p. 392),

ds = an element of the unit sphere s , and

δg = vertical derivative of the disturbing potential, i.e., the
gravity disturbance,

we found formulas for computing the deflections of the vertical as well as their horizontal derivatives using gravity disturbances. In these formulas the kernels

$$H' = \frac{dH(\psi)}{d\psi} \quad \text{and} \quad H'' = \frac{d^2 H(\psi)}{d\psi^2}$$

were used. Because H and H' tend to infinity when the spherical distance (P, ds) tends to zero, it makes sense to verify these relations using a generalization of eq. (1), which corresponds to the Stokes-Pizzetti generalization of Stokes' formula. The corresponding kernel solves the Neumann boundary value problem in the space exterior to the geoid or to the approximating sphere. This closed formula is again found in Hotine (1969, p. 392, eq. 29.16). Denoting by r the radius vector of the point P at which T is computed, we have

$$H(r, \psi) = \frac{2R}{r\sqrt{\Phi}} - \frac{\ln(\sqrt{\Phi} + R/r - \cos\psi)}{1 - \cos\psi} \quad (2)$$

where $\Phi = [1 - 2(R/r) \cos \psi + (R/r)^2]$. Note that $r > R$. The functions $H'(r, \psi)$ and $H''(r, \psi)$ are derived in appendix A for $r = R$; they are identical to the derivatives of

$$H \equiv H(\psi) \quad \text{and} \quad H' \equiv H'(\psi)$$

as discussed by Groten (1979). Because planar approximations are sufficient for computing second derivatives of the disturbing potential, only H' is important for operational geodesy. Analogous to the Vening-Meinesz formula we obtain (Hotine 1969, p. 318, para. 36)

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix}_P = \frac{1}{4\pi\gamma} \iint_S \delta g H'(\psi) \begin{pmatrix} \cos \\ \sin \end{pmatrix} \alpha \sin \psi d\psi d\alpha. \quad (3)$$

where α is azimuth and γ is normal gravity. For the solution of the spatial case we obtain

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix}_P = \frac{R}{4\pi\gamma r} \iint_S \delta g H'(r, \psi) \begin{pmatrix} \cos \\ \sin \end{pmatrix} \alpha \sin \psi d\psi d\alpha. \quad (4)$$

The second derivative $H'' = \partial^2 H / \partial \psi^2$ is of interest for interpolating deflections which, because it is locally applied, is accomplished more easily by collocation.

We can write the analogous least-squares collocation solution in the form of

$$F(P) = \underline{K} \underline{C}^{-1} \underline{f} \quad (5)$$

where $F(P)$ represents (ξ, η) of eq. (4) or T of eq. (1); \underline{f} is the vector of discrete gravity disturbance values and \underline{C} is the sum

$$\underline{\bar{C}} + \underline{D} \quad (6)$$

where $\underline{\bar{C}}$ is the autocovariance matrix; \underline{D} is the error covariance matrix of \underline{f} , assuming zero correlation between noise and signal; and \underline{K} is the cross-covariance matrix between \underline{f} and \underline{F} . Additional corrections to eq. (5) may be necessary depending on how \underline{K} is derived.

By comparing vertical gradient formulas for δg and Δg (e.g., Heiskanen and Moritz (1967: p. 115, formula 2-217) and Molodenskii et al. (1962: formula II.2.5)), it is realized that planar approximations of vertical gradients of Δg and δg are identical. Spherical approximations differ by

$$\frac{2\Delta g(P)}{R} + \frac{6T(P)}{R^2}. \quad (7)$$

Consequently, the downward continuation of gravity disturbances δg is fully analogous to the downward continuation of gravity anomalies. Therefore, we could also apply eq. (5) to the downward continuation problem, i.e., the determination of gravity disturbances on the Bjerhammar sphere from surface gravity disturbances. In any case, the smoothing usually associated with the application of least-squares collocation techniques is necessary in unstable downward continuation processes.

Even though the vertical gradient formula of δg includes the small term $6T(P)/R^2$, which is lacking in the formula for the anomalous vertical gradient, the first-mentioned formula is basically less intricate than the latter. In δg , both actual gravity and normal gravity are referred to the same level. Therefore, in the vertical gradient the variation (with height) in the separations of level surfaces of actual gravity from those of normal gravity does not play any role; whereas, in the case of the anomalous gravity gradient the increase of separation of "geops" from "spherops" of the analytically continued external potential (which is different from the internal potential) complicates the computation with increasing depth.

Concerning the difference between gravity disturbances on the sphere and at the Earth's surface S , we must be careful when deriving the cross-covariance from experimental data using covariance propagation. Whenever the Vening-Meinesz kernel (in the case of Δg) and the kernel H' (in the case of δg) are used for the derivation of \underline{K} in determining (ξ, η) , then Δg or δg referred to the sphere will yield an approximation that corresponds to the classical boundary value problem, i.e., a Stokes-type approximation. According to the gradient solution of the oblique-derivative boundary value problem, we may replace δg with

$$\delta g - \frac{\partial \delta g}{\partial h} h \quad \text{or} \quad \delta g - \frac{\partial \delta g}{\partial h} [h - h(P)] \quad (8)$$

where h is the elevation of the "running point" on S .

In other words, for practical applications Bjerhammar's and Molodensky's concepts can be transferred from the Δg solutions to the δg solutions. For very precise computations, of course, we have to be aware of the differing definitions of (ξ, η) on the geoid and at the Earth's surface. Moreover, the separation of the geopotential (geop) surface from the corresponding spherical potential (spherop) surface depends on the elevation; therefore, it should be noted whether the point of evaluation, P , is on the geoid or in space. Consequently, we arrive at a variety of possible mixtures of collocation solutions with conventional procedures. The chosen mixture depends strongly on the desired degree of smoothness emerging from the final results. The degree of smoothness depends on the specific covariance functions used in the application. The impact of this choice is seen in various examples of practical applications, such as those used by Becker (1980).

3. TRUNCATION ERROR

In a previous investigation (Groten 1979), the original function

$$H = \sum_{n=0}^{\infty} H_n = \sum_{n=0}^{\infty} \frac{2n+1}{n+1} P_n(\cos\psi) \quad (9)$$

was replaced by

$$\bar{H} = \sum_{n=2}^{\infty} \frac{2n+1}{n+1} P_n(\cos\psi). \quad (10)$$

With this function the solutions are referred to a geocentric location and the "scale problem" associated with H_0 is separated from the problem itself, as in case of the Stokesian solution. Since

$$H - \bar{H} = 1 + \frac{3}{2} \cos \psi \quad (11)$$

it is readily seen that the behavior of the kernel H depends on whether or not the first two spherical harmonics are included in such formulas as eqs. (1) or (3). This behavior affects the influence of the remote zones in those integral formulas. Because Groten (1979) previously found that the behavior of eq. (9) is significantly superior to eq. (10) when remote zones are to be considered (we associate the term "truncation error" with this omission in general), the modified function \bar{H} is only mentioned here.

Moreover, for autocovariance functions, as used in least-squares collocation, we always begin the summation with $n = 3$ for practical reasons. Therefore, the functions H and \bar{H} need not be distinguished in these approaches. In principle, we could start with δg from $n = 1$ for the summation of degree variances.

Since the data at hand are deficient, we need to apply methods in which the influence of presently available gravity models is minimized. In areas like North America and Europe, for example, we have at our disposal a reasonable field in the neighborhood of a station. For the remote zones a truncated model, such as the Goddard Space Flight Center Model GEM 10B, is available. The inadequacies of present global models are well known. The qualitative comparison between Stokes' type of solution and Neumann's type of solution was given by Groten (1979). This was supplemented by Stock (1980) who made quantitative comparisons for specific gravity field parameters referred to geoid undulations. Analogous studies for specific fields

related to (ξ, η) are planned in a forthcoming paper by Stock. These quantitative investigations fully corroborate the conceptual discussion given by Groten (1979), and are all based on the error integral introduced by Groten and Moritz (1964):

$$\int \bar{k}^{-2} (\psi) \sin \psi \, d\psi. \quad (12)$$

These have also recently been applied by Ihde (1980), where \bar{k} denotes the kernel; i.e., \bar{k} takes the place of H or H' .

For least-squares collocation, we can draw the important conclusion from these results that, to some extent, the high pass filtering involved in the transition from

$$\delta g = \sum_{n=2}^{\infty} \delta g_n \quad (13)$$

to

$$\Delta g = \sum_{n=2}^{\infty} \Delta g_n \quad (14)$$

is compensated by the substitution of the Stokes kernel

$$S = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n (\cos \psi) \quad (15)$$

by

$$H = \sum_{n=0}^{\infty} \frac{2n+1}{n+1} P_n (\cos \psi). \quad (16)$$

(The fact that S is used for the Earth's surface as well as for Stokes' kernel should not be confusing.)

Consequently, the autocovariance function of δg on the unit sphere

$$\text{cov}(\delta g, \delta g) = \sum_{n=3}^{\infty} \sigma_n(\delta g) P_n(\cos \psi), \quad (17)$$

where $\sigma_n(\delta g)$ is the degree variances of δg , does not necessarily imply that the influence of the remote zone increases when compared to

$$\text{cov}(\Delta g, \Delta g) = \sum_{n=3}^{\infty} \sigma_n(\Delta g) P_n(\cos \psi). \quad (18)$$

Here

$$\sigma_n(\Delta g) = \sigma_n(\delta g) \left(\frac{n-1}{n+1} \right)^2 \quad (19)$$

is a straightforward consequence of

$$T_n = R \frac{\Delta g_n}{n-1} = R \frac{\delta g_n}{n+1}. \quad (20)$$

Because $(n-1)/(n+1)$ as well as $(n-1)^2/(n+1)^2$ tend sufficiently close to 1 as n goes to 30, the presently available gravity models seem to be satisfactory for a comparison of the high and low pass filtering effects inherent in the transitions $S \rightarrow H$ and $\Delta g \rightarrow \delta g$, respectively. Also, Stock (1980) introduced assumptions that might be considered as one-sided; hence, a further investigation of these assumptions is necessary.

The practical consequences of the relation shown in eq. (20) become obvious when we consider the power spectrum within the range of

$$3 \leq n \leq 30$$

using the harmonics of the GEM 10 model. Tables 1a and b compare the sums

Table 1a--Power spectrum constituents corresponding to GEM 10 for δg and Δg .
For σ_{19} , instead of 2.8 from GEM 10, the value 2.0 was used.

u	A (mgal ²)	A' (mgal ²)	B (mgal ²)	B' (mgal ²)
3	412.5	134.0	187.0	33.5
4	278.5	188.4	153.5	53.1
5	224.1	234.8	133.9	73.7
6	177.8	272.0	113.3	92.7
7	140.5	306.0	94.3	111.8
8	106.6	324.8	75.2	123.2
9	87.7	342.1	63.8	134.3
10	70.4	356.6	52.7	144.0
11	55.9	366.1	43.0	150.6
12	46.4	371.1	36.4	154.2
13	41.4	379.5	32.8	160.4
14	33.0	384.0	26.6	163.8
15	28.5	387.9	23.2	166.8
16	24.6	391.2	20.2	169.4
17	21.3	393.8	17.6	171.5
18	18.7	397.7	15.5	174.6
19	14.8	400.2	12.4	176.6
20	12.3	402.6	10.4	178.6
21	9.9	404.8	8.4	180.4
22	7.7	406.8	6.6	182.1
23	5.7	407.4	4.9	182.6
24	5.1	407.9	4.4	183.0
25	4.6	408.7	4.0	183.7
26	3.8	408.7	3.3	183.7
27	3.8	409.4	3.3	184.3
28	3.1	410.4	2.7	185.2
29	2.1	411.1	1.8	185.8
30	1.4	412.5	1.2	187.0

Table 1b--Power spectrum constituents related to a sphere

u	A (mgal ²)	A' (mgal ²)	B (mgal ²)	B' (mgal ²)
3	420.6	397.6	191.2	178.6
4	285.0	266.4	157.3	145.8
5	229.8	213.3	137.4	126.7
6	182.7	168.3	116.5	106.7
7	144.9	132.4	97.2	88.4
8	110.2	99.1	77.7	70.1
9	90.9	81.9	66.0	59.2
10	73.1	65.3	54.6	48.6
11	58.2	51.6	44.6	39.4
12	48.4	42.7	37.8	33.2
13	43.2	38.0	34.1	29.8
14	34.5	30.0	27.7	24.0
15	29.8	25.7	24.2	20.8
16	25.8	22.0	21.1	18.0
17	22.3	18.9	18.4	15.6
18	19.5	16.5	16.2	13.7
19	15.5	13.0	13.0	10.9
20	12.9	10.8	10.9	9.1
21	10.4	8.6	8.8	7.3
22	8.1	6.7	6.9	5.7
23	5.9	4.9	5.1	4.2
24	5.3	4.4	4.6	3.8
25	4.8	3.9	4.2	3.4
26	4.0	3.2	3.5	2.8
27	4.0	3.2	3.5	2.8
28	3.3	2.6	2.9	2.3
29	2.2	1.7	1.9	1.5
30	1.5	1.1	1.3	1.0

$$A = \sum_{n=u}^{n=30} \sigma_n(\delta g)$$

$$B = \sum_{n=u}^{n=30} \sigma_n(\Delta g)$$

$$A' = \sum_{n=3}^{n=u} \sigma_n(\delta g) \quad (21)$$

and

$$B' = \sum_{n=3}^{n=u} \sigma_n(\Delta g) \quad (22)$$

for $3 \leq u \leq 30$. A, A', B, and B' refer to the GEM mean equatorial radius $a=6378140$ m. Thus, the part of the power inherent in specific partial sums can immediately be seen starting from $u = 3$ as well as from $u = 30$. For $u > 30$, the uncertainty of the coefficient is substantially higher than the difference between $\sigma_n(\delta g)$ and $\sigma_n(\Delta g)$.

We can conclude from table 1a that it may be advantageous to apply collocation procedures to gravity disturbance residuals

$$\sum_{n=10}^{\infty} \delta g_n$$

instead of to the gravity disturbances themselves. The low harmonics portion can be solved by using spherical harmonic formulas, such as eq. (20), or corresponding integral formulas. This conclusion is corroborated by considering the partial sums in table 1b.

To evaluate eq. (17) the degree variances have to be transformed from the GEM equatorial radius "a" to a mean sphere of radius $R_E = 6371$ km. The corresponding transformed degree variance sums are shown in table 1b.

Stock (1980) discusses in detail the gravity field characteristics upon which his results are based. A less specific and more generally valid representation is obtained by forming the ratios of these truncation errors. These are basically found in the integral of eq. (12) in the form

$$\int_{\psi_0}^{\pi} H^2(\psi) \sin \psi \, d\psi \quad (23)$$

for the potential and the geoid undulation, and they depend on the error variances and covariances of the mean gravity values in the zone $\psi_0 \leq \psi \leq \pi$ of the integral eq. (1) and on the size of that zone. Because the error covariances are neglected in the above solution, which limits the validity of the numerical solutions, the ratios are definitely preferable. Consequently, figure 1 shows the ratio of truncation errors, e , i.e.,

$$F = \frac{e(\delta g)}{e(\Delta g)} \quad (24)$$

where the error variances and error covariances of δg and Δg can be considered to be nearly identical because the error contribution of T in

$$\delta g = \Delta g - \frac{\partial \gamma}{\partial h} N = + \Delta g + \frac{2T}{R}, \quad (25)$$

where h is elevation, may be neglected. Nevertheless, this error source has been taken into account.

As far as eq. (25) is concerned, some care is necessary. Heiskanen and Moritz (1967: p. 85) define δg as the negative gradient $-\partial T/\partial h$, whereas Hotine (1969: p. 392, formula 29.25) applies the positive gradient $\partial T/\partial h$. Analogously, δg is defined by Heiskanen and Moritz (1967: p. 85), which is in agreement with our eqs. (20) and (25), whereas Hotine (1969, p. 392 formulas 29.30 and 29.33) uses the opposite sign.

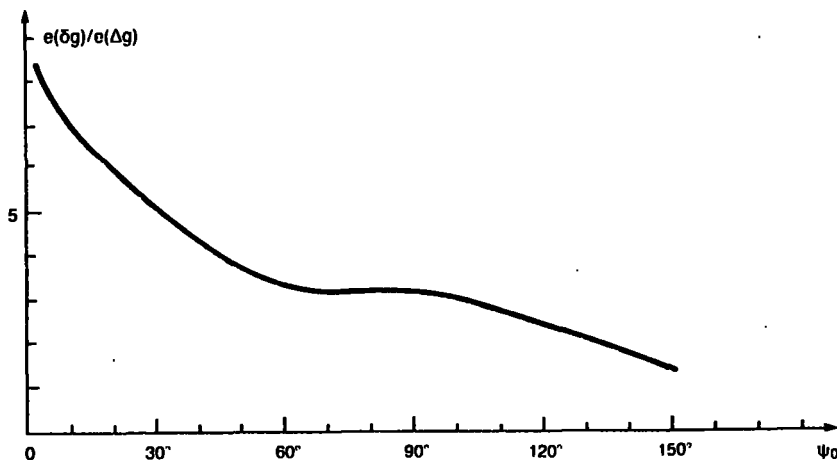


Figure 1.--Truncation error ratios as a function of ψ_0 .

An apparent conclusion we may draw from tables 1a and b is

$$|\Delta g| < |\delta g|.$$

However, in areas such as the Mediterranean, where $\Delta g < 0$ and $N > 0$ often occur together, we can have

$$|\delta g| < |\Delta g|$$

because of the term

$$\frac{\partial \gamma}{\partial h} N \doteq -0.3086N \quad (\text{in mgal/m})$$

(with geoid heights N in meters), which represents the difference between gravity anomalies and gravity disturbances in the sense of $\Delta g - \delta g$. Because Stokes' function as well as Hotine's function H tends to infinity as $\psi \rightarrow 0$ (i.e., in the nearest neighborhood around the point of computation) the influence of that zone is overwhelming. Therefore, if $|\delta g| < |\Delta g|$ in that area, the integral of eq. (1) has a corresponding advantage over Stokes' integral.

An alternative representation of $T(\delta g)$ can be deduced from Molodenskii et al. (1962, p. 147). A straightforward analogy using derivations of Molodenskii immediately yields

$$T = \frac{R}{4\pi} \int_0^{\psi_0} \int_0^{2\pi} \delta g H \, ds + \frac{R}{2} \sum_0^{\infty} q_n(\psi_0) \delta g_n \quad (26)$$

where $q_n(\psi_0)$ are truncation coefficients that have been computed in detail by Groten and Jochemczyk (1978).

Analogously,

$$\begin{aligned}
 \begin{pmatrix} \xi \\ \eta \end{pmatrix} &= \frac{1}{4\pi\gamma} \int_0^{\psi_0} \int_0^{2\pi} \delta g H' \begin{pmatrix} \cos \\ \sin \end{pmatrix} \alpha \, d\alpha \sin \psi \, d\psi \\
 &- \frac{H(\cos \psi_0) \sin \psi_0}{4\pi\gamma} \int_0^{2\pi} \delta g \begin{pmatrix} \cos \\ \sin \end{pmatrix} \alpha \, d\alpha \\
 &- \frac{1}{2\gamma} \sum_{n=2}^{\infty} q_n \frac{\partial \delta g_n}{\left\{ \frac{\partial \phi}{\cos \phi} \frac{\partial \lambda}{\partial \lambda} \right\}}.
 \end{aligned} \tag{27}$$

Just as the integral of the type shown in eq. (23) depends on the integral in eq. (26) and the integral

$$\int_{\psi_0}^{\pi} [H'(\psi)]^2 \sin \psi \, d\psi \tag{28}$$

depends on the first integral in eq. (22), the average contribution represented by the series in eq. (26) can be given by an analogous series already used by several authors, which was derived from a formula of Molodenskii et al. (1962, p. 164).

The derivation of the average total deflection of the vertical

$$\overline{\delta \theta^2} = \frac{1}{4\gamma^2} \sum n(n+1) q_n^2 (\psi_0) \sigma_n (\delta g) \tag{29}$$

is straightforward. Here the total deflection is given by

$$\theta^2 = \xi^2 + \eta^2. \tag{30}$$

The analogous relation for Vening-Meinesz's formula reads (Molodenskii et al. 1962, p. 166)

$$\overline{\delta\theta^2} = \frac{1}{4\gamma^2} \sum n(n+1) Q_n^2(\psi_0) \sigma_n(\Delta g). \quad (31)$$

The formulas for some Q_n are given in analytical form by Molodenskii et al. (1962, p. 148). However, more appropriate high speed computer-oriented recursion formulas have been published by various authors, which are well known.

Differencing, from formulas (29) and (31) we obtain

$$\frac{1}{4\gamma^2} \sum n(n+1) \sigma_n(\Delta g) \left[Q_n^2 - q_n^2 \left(\frac{n+1}{n-1} \right)^2 \right]. \quad (32)$$

Setting $\psi_0 = 0$ yields (Molodenskii et al. 1962, p. 87)

$$\overline{\theta^2} = \frac{1}{\gamma^2} \sum \frac{n(n+1)}{(n-1)^2} \sigma_n(\Delta g) \quad (32a)$$

as a consequence of

$$Q_n(\psi_0) = \frac{2}{n-1} \quad (33)$$

for $\psi_0 = 0$. From Groten and Jochemczyk (1978), we obtain

$$q_n(\psi=0) = \frac{2}{n+1}. \quad (34)$$

For $\psi = 0$, eq. (32) must vanish. By inserting eqs. (34) and (33) into eq. (32) we obtain a check on eq. (32).

4. ALTERNATIVE GRAVITY FIELD REPRESENTATIONS

Contrary to the single layer density in the oblique boundary value approach, which involves intricate mathematical handling, the use of single layer densities on a sphere is easy and efficient. Even though its advantages are similar to the gravity disturbances as far as the truncation error is concerned, it will be considered in the present context for its high harmonic gravity field representation. In this case Δg and N have to be continued downward onto the Bjerhammar sphere, yielding

$$\mu = \Delta g + \frac{3\gamma}{2R} N \quad (35)$$

$$\mu = \Delta g + 0.2311 N \quad (36)$$

where Δg is given in milligals and N in meters. Using eq. (25) we find

$$\mu = \delta g - 0.0770 N \quad (37)$$

and

$$\mu = \delta g - \frac{\gamma}{2R} N . \quad (38)$$

The analytical continuation of

$$N = T/\gamma \quad (39)$$

is easily performed by applying (Heiskanen and Moritz 1967, p. 310)

$$\frac{\partial N}{\partial h} = - \frac{\Delta g}{\gamma} . \quad (40)$$

This type of analytical continuation from the surface to the sphere at the Earth's interior can also be made by collocation. However, first derivatives of the potential are continuous even in the interior of the Earth. In contrast, second derivatives are discontinuous, in general, so that the aforementioned rather sophisticated downward procedures are more appropriate for Δg than for elementary gradient formulas.

By taking a surface density of a single layer form, such as

$$\text{or} \quad 2\pi G\sigma = \mu \quad (41)$$

$$\sigma = \frac{\mu}{2\pi G} , \quad (42)$$

where G is the Newtonian gravitational constant, we conclude that gravity disturbances are capable of providing a good approximation for such a surface density. This result is important for a representation of the high harmonic part of the gravity field. It is realized that δg is a useful substitute for μ in various approximations, e.g., in error estimation and feasibility studies.

The use of single-layer density formulas, such as

$$T(P) = \frac{R^2}{2\pi} \iint \frac{\mu}{\ell} ds \quad (43)$$

applied to a sphere (with ℓ being the straight line distance between P and ds) for determinations of (N, ξ, η) , is discussed by Groten (1979) and Stock (1980). Details on truncation error behavior can be found in these two sources.

5. A NEW TYPE OF GEODETIC BOUNDARY VALUE PROBLEM

Whenever the geocentric coordinates of any station $P(\vec{r})$ are given, the determination of the corresponding ellipsoidal coordinates (B, L, H) (with respect to any arbitrary geocentric reference ellipsoid) is a simple algebraic transformation that is solved by a one- or two-step iteration procedure. Consequently, whenever gravity is given at P we can evaluate normal gravity $\gamma(P)$ using $P(B, L, H)$. Moritz (1974) has applied this reasoning to satellite altimetry for sea surface topography determination. Similarly, on land $\gamma(P)$ can be evaluated if geocentric station coordinates are obtained from tracking Global Positioning System satellites or similar types of satellites. In principle, it does not matter whether or not g is obtained from GRAVSAT-type satellites or from surface gravity. In the case of precise satellite information on the gravity potential, we obtain N , or height anomalies ξ , immediately from eq. (39), assuming that the terrestrial gravitational constant GM , as well as a "scale quantity" (such as the semi-major axis of the Earth ellipsoid), is precisely known. With Very Long Baseline Interferometry (VLBI) and other high-precision techniques now available the "scale" is no longer taken from such equations as (Heiskanen and Moritz 1967: pp. 101-103)

$$N_0 = \frac{\delta(GM)}{R\gamma} - \frac{\delta W}{\gamma} \quad (44)$$

where N_0 is the zero degree harmonic of geoid height N , $\delta(GM)$ is the error in GM , and $\delta W = W^0 - U^0$ is the difference of the actual potential W on the geoid and the normal potential U on the ellipsoid. Based on the solution of the boundary value problem at the Earth's surface we can derive a corresponding gravity $\delta W' = \delta W$ that represents the difference between the actual potential at P and the normal potential on the ellipsoid. Again eq. (40) can be used. Thus $\delta W' - \delta W$ is readily computed.

With the present uncertainty in the value of GM we arrive at a corresponding offset N_0 on the order of

$$N_0 \doteq 0.3 \text{ m.}$$

Thus N is given either by

$$T = W - U, \quad (45)$$

together with eq. (39), or by a formula such as eq. (1), which uses δg . The deflections of the vertical are obtained by using eqs. (4) or (5) or similar formulas. Because of the strong influence of high harmonics in (ξ, η) , as seen by inspecting formulas such as Groten's eq. (6.229) (Groten 1979/80, p. 488), the direct use of differentials in eq. (45) is not possible. On the other hand, the harmonics derived from satellites can be used in eq. (27) together with terrestrial data. However, because astronomical coordinates are no longer of primary importance in modern geodesy we can omit details.

The difference between geoid undulation, or height anomaly, and ellipsoidal height subsequently yields the orthometric height, h , or normal height, \bar{h} , respectively, i.e.,

$$h = H - N^2 \quad (46)$$

$$\bar{h} = H - \xi. \quad (47)$$

By comparing orthometric heights h and normal heights \bar{h} obtained in this manner with terrestrial results we can determine the offset of the national height system. "Offset" means the difference between the conventional "zero point" and the geoid at the fundamental station of the vertical datum. Obviously,

$$u = \bar{h} - h = N - \xi \quad (48)$$

where u is the separation between the geoid and the quasi-geoid. (Both surfaces coincide, of course, on the oceans, i.e., at a tide gage representing a zero level of the leveling systems). Using this formula, we can convert classical geoidal systems into modern Molodensky-type systems. Useful operational formulas based on eq. (40) can be found in conventional textbooks. On the other hand, from the definition of normal and orthometric heights we immediately obtain

$$u = (\bar{g} - \gamma) h / \bar{\gamma} \quad (49)$$

²The use of H for Hotine's function as well as for the ellipsoidal heights should not be confused.

where the bar indicates mean values taken along the plumb lines between the respective reference surfaces.

Since the difference between ground disturbances $\delta g(P)$ and sea level disturbances referred to the geoid, or to a Bjerhammar sphere, was considered by taking into account the terms in eq. (8), we thus obtain a complete system that enables us to compare leveling results with elevation data resulting from eqs. (46) and (47). Since Bjerhammar's solution of the boundary value problem gives results that are referred to boundary values on an exact sphere, there is no spherical approximation involved whenever eqs. (46) and (47) are applied in the exact form.

Consequently, we have the possibility of verifying distortions in leveling results by using space techniques applied to physical geodesy. Holdahl (1981) and others have described the principal sources of such distortions.

In view of such errors and other difficulties especially associated with the implementation of orthometric heights, a sufficient approximation for eq. (48) in many cases is given by Heiskanen and Moritz (1967, p. 328)

$$N - \xi \doteq \Delta g_B h , \quad (50)$$

where the left side is in meters, if Δg_B (Bouguer anomaly) is given in gals and h is in kilometers.

These determinations of the reference systems upon which first-order levelings are based are of special interest if the zero point of the system is no longer available, as in the case of several national leveling systems in Europe. There is, for example, no access to the fundamental base station in the German leveling system; moreover, the tide gage representing the zero level no longer exists. Consequently, the offset of the network itself with respect to the geoid or to the quasi-geoid has to be determined without reference to the physical "zero level." A more detailed discussion of the "zero reference" follows.

6. RELATIVE COMPARISON OF SPACE AND TERRESTRIAL DATA

Contrary to absolute determinations of ellipsoidal heights

$$H = N + h ,$$

as obtained from space and terrestrial data, the N_0 term does not play any role in comparing differences of ellipsoidal heights at different stations. If very high accuracy is desired, the ephemeris errors limit the extent of the area where data can be used to determine relative

locations (including elevation differences) with an accuracy of a few centimeters. Although comparisons of distance measurements (satellite laser observations vs. VLBI) indicate an accuracy of a few centimeters even over very long distances, the absolute orientation of relative station location vectors is not yet available with comparable accuracy. However, over short distances, $d < 100$ km or so, an accuracy of about 2 cm is expected from VLBI techniques applied to the Global Positioning System.

If, instead of using the "exact sphere" approach given by Bjerhammar, the conventional spherical approximation is applied in determining the geoid and plumb line deflection, corresponding ellipsoidal corrections may offset distances $d \gg 100$ km. Mather (1973) considered that correction (other formulas are well known from other studies by Bjerhammar, Lelgemann, Zagrebin, and others) together with atmospheric and similar corrections which must be considered in absolute comparisons. When one aims at accuracies of better than ± 20 cm, then these corrections must be incorporated. Formulas that were derived for use in combination with Δg can also be directly applied to δg . However, the indirect effects associated with the atmospheric mass shift can always be neglected. These amount to less than 1 cm. (See, e.g., Moritz (1980, p. 425.)

Consequently, relative comparisons of the vertical station coordinates obtained from space data with those obtained from applications of physical geodetic techniques do not involve theoretical difficulties if the distances are of the order of 100 km. (For details on systematic distortions in leveling, see, e.g., Vaníček et al. (1980).)

7. REMARKS ON COMBINING SPACE AND TERRESTRIAL DATA

It might make sense to consider the mixed boundary value problem where gravity is given on some part of the Earth's surface S and the disturbing potential

$$T = N\gamma \tag{39a}$$

is supposed to be obtained from satellite altimetry for the remaining part of S . The corresponding mixed boundary value problem involves difficulties if it is applied to $(\Delta g, T)_S$ in the determination of $T(P)$, where P is again a point in space exterior to S . The problem is much easier to handle when gravity is again given in terms of disturbances δg . The mixed problem, which is a combination of a Neumannian and a Dirichlet problem referred to a Bjerhammar sphere, i.e.,

$$(\delta g, T)_S \longrightarrow T(P) \tag{51}$$

has been covered in the mathematical literature since 1933. (See, e.g., Giraud (1933).)

However, when considering an altimeter resolution on the order of ± 10 cm or better, a serious problem can arise: according to the recommendation passed by the International Association of Geodesy at the General Assembly of the International Union of Geodesy and Geophysics in Canberra, 1980, the total tidal effect has to be eliminated in geodetic data. To eliminate the so-called "permanent tide" in satellite altimetry data, it would be necessary to know the distribution of the relevant deformation parameters in terms of the Love number k , although the density distribution of the Earth is not needed. Since the secular k is not available, we cannot reduce the actual mean sea surface to a tide-free model of the sea surface. We can only reduce it to an arbitrarily defined surface by using two doubtful second-degree Love numbers (h, k). In principle, a reduction of all data is possible by using the same tide-free Earth model. But models available today are not necessarily "tide free."

Moreover, the equipotential surfaces that correspond to an arbitrary Earth model would be meaningless from the viewpoint of geophysical interpretation as far as ocean streams and currents are concerned. (For further details on permanent tide problems, see Groten (1979).)

8. THE BASIC ZERO-REFERENCE OF HEIGHT SYSTEMS

P. Vaníček et al. (1980) reviewed the arguments in favor of a relatively stable geoid, one that is only weakly affected by elevation and gravity variations with time. On the other hand, at sea and along the coast, the situation obviously is more intricate and complicated, involving the previously mentioned difficulties to define a unified zero level of height systems. Mean sea level, which defines the volume of the Earth, can be considered, to some extent, as one part of the four defining quantities of the normal gravity potential if it replaces, for example, the semimajor axis of the ellipsoid. Consequently, for precise geodesy we need a rigorous definition for mean sea level. At the present time, this problem and other related questions have not been solved, so that emphasis must be put on relative comparisons, as pointed out in section 6, instead of absolute comparisons of space data with terrestrial results, as applied to techniques of physical geodesy. Therefore, any discussion of nonlinear solutions for the geodetic boundary value problem seems to be premature as far as their possible application is concerned.

Moreover, gravity disturbances of surface type $\delta g(P)$, as well as gravity disturbances at the geoid $\delta g(G)$, are affected by offsets and systematic distortions of the height system that is used for evaluating normal gravity $\gamma(P)$ and gravity $g(G)$ on the geoid:

$$\delta g(P) = g(P) - \left(\gamma(E) - \frac{\partial \gamma}{\partial h} H \right)$$

$$\delta g(G) = g(P) + \frac{\partial g}{\partial h} - \left(\gamma(E) - \frac{\partial \gamma}{\partial h} N \right).$$

Consequently, with $\gamma(E) = \gamma$ on the ellipsoid and

$$\frac{\partial g}{\partial h} \doteq \frac{\partial \gamma}{\partial h} \doteq 0.3086 \text{ mgal/m}$$

we obtain the approximation

$$\delta g(P) \doteq \delta g(G).$$

One of the most serious problems in the precise application of physical geodesy is shown in eq. (46), where eq. (45) must be taken into account as well as the offset of the leveling systems. Therefore, to be more rigorous, eq. (47) should be written as

$$H = \zeta + \zeta_0 + \bar{h} + \delta\bar{h}$$

where ζ_0 is the height anomaly harmonic of zero degree; $\delta\bar{h} \doteq \delta h$ is the offset of the specific height system caused by the difference between the local sea surface topography at the zero reference point of the leveling system, and the height of the quasi-geoid is the geoid height at the tide gage station. This offset has about the same value whether it is expressed in the orthometric system or the normal height system.

The problem of a unifying local height system can be solved approximately when a precise global gravity field becomes available by using GRAVSAT-type data or by global station positioning with VLBI and/or satellite laser positioning. Rizos (1980) has shown how difficult it is to define a precise geoid (which is the zero reference for elevations) with an accuracy on the order of a few centimeters. On the other hand, there is no need for such a precisely defined geoid if we can measure or model the instantaneous elevations above the ellipsoid at a specific epoch on land and at sea. This is basically possible when relative station coordinates are obtained from VLBI measurements or with lower accuracy from Doppler data. Absolute coordinates are obtained from satellite laser tracking; at sea, precise altimetry enables the determination of coordinates with corresponding accuracy. Therefore, a unified worldwide height reference system is feasible by tying together the different vertical datums. All we basically need to know is the volume of the Earth and the terrestrial gravitational constant GM ; for the volume, the potential W^0 at the sea surface can be substituted in principle, but then the definition of the sea surface enters again.

If the gravity values are referred to a constant speed and axis of rotation $\bar{\omega}$, (implying mostly negligible corrections for polar motion and a ω variable only to a few microgals), then the only question remaining to be solved is the flattening to which the data should be referred. The tidal potential has to be completely removed from all geodetic data in order to obtain a harmonic disturbing potential because, being an "internal potential," the tidal potential U' itself is nonharmonic. In other words, the tidal potential does not go to zero with increasing distance from the geocenter. On the other hand, the permanent tides M_0 and S_0 cannot be removed completely. That is, the direct part of U' related to M_0 and S_0 can be eliminated, but the remaining part $U' (k-h)$ of

$$U' (1 + k - h)$$

cannot be separated because the associated second-degree Love numbers (h , k) are not known. In addition, the part of the deformation of the sea surface $U' (1+k)/g$ caused by M_0 and S_0 has to be eliminated. Therefore, whenever sea altimetry is involved, it must also be reduced. However, $U' (k-h)$ is of a different nature; i.e., it can be represented by an external potential. Consequently, its removal is not necessary if we refer all data to the actual surface, i.e., the flattening f corresponding to the actual J_2 of the Earth. This is done by just eliminating U' together with the transient part of $U' (1+k-h)$ from all geodetic data.

If we removed the permanent constituents of $U' (1+k-h)$ instead of $U' (M_0, S_0)$, then the purely geometrical data would also be affected. Secular (h , k) are unknown. We know only that such crude estimates as $k=0.95$ are derived from doubtful quantities. Even less is known about sufficient accuracy and little is known about the permanent tidal deformation of the sea surface. Therefore, we should apply the tidal correction in such a way that only the smallest number of hypotheses is introduced. Consequently, we should remove those parts that must be eliminated to obtain a harmonic disturbing potential. When $U' (M_0, S_0)$ is removed, together with the transient parts of $U' (1+k-h)$, a solution is achieved that is practically free of hypotheses. This statement is true if geometrical observations (such as those employing VLBI data) are correctly reduced for transient Earth tide constituents. This method is feasible with sufficient data at hand.

Thus, the basic questions related to a normal gravity potential are solved, with f , GM , ω , and the semimajor axis or volume of the ellipsoid being the four main parameters. A few additional remarks are in order for high-precision geodesy:

When normal gravity is needed with an accuracy better than $\pm 70 \mu\text{gal}$, then formulas such as

$$\gamma_0 = \gamma_e (1 + \beta_1 \sin^2 \phi + \beta_2 \sin^2 2\phi)$$

are adequate. Whenever an accuracy of $\pm 5 \mu\text{gal}$ is important, then

$$\gamma_0 = \gamma_e (1 + \beta_3 \sin^2 \phi + \beta_4 \sin^4 \phi)$$

is preferred. Rizos (1980, p. 172) carefully studied this topic. Numerical values for the coefficients β_i ($i=1,2,3,4$) are found in the special issue of the Bulletin Geodesique (Geodetic Reference System 1967, International Association of Geodesy, Paris, 1970) for the 1971 reference system. By γ_0 we denote gravity on the ellipsoid; $\gamma_e = \gamma$ on the ellipsoid at the latitude $\phi = 0$, i.e., at the equator.

Using eq. (20) we obtain

$$N_n = \frac{R \Delta g_n}{\gamma(n-1)} = \frac{R \delta g_n}{\gamma(n+1)} \quad (20a)$$

for the spherical harmonic expansion

$$N = \sum N_n$$

of geoid undulations. Using $n=2$ and $n=33$, respectively, we realize that if such a gravity harmonic is known to an accuracy of $\pm 50 \mu\text{gal}$ we will obtain geoid height harmonics to an accuracy of $\pm 30 \text{ cm}$ and $\pm 1 \text{ cm}$ or better. From the well-known Taylor expansion of normal gravity at elevation H above the ellipsoid E

$$\gamma(H) = \gamma(E) + \frac{\partial \gamma}{\partial h} h + \frac{1}{2} \frac{\partial^2 \gamma}{\partial h^2} h^2 + \dots$$

with (Heiskanen and Moritz 1967, p. 78)

$$\frac{\partial \gamma}{\partial h} = - \frac{2\gamma}{a} (1 + f + m - 2f \sin^2 \phi)$$

and

$$\frac{\partial^2 \gamma}{\partial h^2} = \frac{6\gamma}{a^2}$$

We obtain approximations that are limited to linear terms in the flattening f . (For a definition of the well-known parameters in these two formulas, see Heiskanen and Moritz (1967).) When applied to the normal gravity systems of 1971 or 1980 Somigliana's formula is capable of yielding normal gravity to an accuracy of better than $\pm 50 \mu\text{gal}$. Uncertainties resulting from atmospheric effects amount to $\pm 5 \text{ cm}$ in geocentric positions (Rizos 1980). These are due to displacement of the mass center and to an uncertainty of $\pm 40 \mu\text{gal}$ caused by temporal shifts in atmospheric masses in the gravity itself. (See, e.g., Christodoulidis 1979.) The relatively small effect of local or regional (high-frequency part) distortions in gravity compared to global or large-scale distortions (low-frequency part) is realized by inspecting eq. (20a). Consequently, absolute determinations of geoid heights are generally more distorted than relative geoid determinations. This is especially true because low harmonics are determined by using satellite orbit analysis. The luni-solar deformation of the Earth causes its principal axis of inertia to migrate around the celestial pole along a spherical cone with a diameter of 2" and with the cone's apex being at the Earth's center of mass. The period of this motion is nearly diurnal. The radius of the circular path around the pole amounts to more than 60 m at an elevation of 200 km (which is close to the anticipated altitude of the GRAVSAT satellite orbit). The resulting errors in the gravity field affect mainly the large scale (global or absolute geoid) determinations. With tidal data at hand, the corresponding influences on the gravity field can be taken into account with an accuracy of better than 50 μgal .

Recent studies by C. C. Goad (1980) at NOAA/NOS National Geodetic Survey reveal astonishingly good agreement of modeled Earth-tide perturbations with observed Earth-tide variations and associated indirect effects. Moreover, forthcoming sea tide models of even longer period tides (E.W. Schwidersky, Naval Surface Weapons Center, Dahlgren, Va., private communication 1980) will soon be available. Therefore, as soon as GRAVSAT-type satellites are available, yielding an accuracy on the order of ± 2 to 3 mgal for mean anomalies of 1° -by- 1° blocks, it is logical to assume that the combined satellite and terrestrial gravity fields will also have an accuracy of at least 3 mgal.

Using the free-air gravity gradient it is readily seen that a $\pm 10 \text{ cm}$ uncertainty implies a corresponding uncertainty of $\pm 30 \mu\text{gal}$ in gravity. Even the present uncertainty of $\pm 30 \text{ cm}$ caused by the uncertainty of the terrestrial gravitational constant GM would imply an uncertainty of only $\pm 90 \mu\text{gal}$ in gravity. This effect again concerns only the zero harmonic, i.e., the absolute determination of geoid heights. With relative accuracy of the same order of magnitude in Doppler location determination (Anderle 1979), but basically higher accuracy in ARIES-type VLBI measurements and substantially higher accuracy anticipated in small-scale VLBI and GPS approaches (MacDoran 1979; Counselman and Shapiro 1979), we can expect that δg will be determined in the future to the same accuracy as Δg . This statement is true for measurements on land; at sea δg is better than Δg because of the deviations of the sea surface from the geoid. The latter will become important as soon as satellites equipped with high-precision altimeters are available.

9. SOME NUMERICAL ASPECTS

Correlation Length Differences

From tables 1a and b, we obtain the relevant difference between the autocorrelation functions

$$\text{cov}(\delta g, \delta g) \text{ and } \text{cov}(\Delta g, \Delta g).$$

Since the higher harmonics have about the same order of magnitude for δg and Δg , we omitted them and plotted (fig. 2) the sum of the harmonics

$$\sum_{n=3}^{30} \sigma_n \quad (52)$$

of degree variances for δg and Δg . (Here again we could begin with $n=1$ instead of $n=3$ in the case of δg .) The difference in correlation length is

$$\delta \ell = \ell_{\Delta g} - \ell_{\delta g} = 10.5^\circ - 13.5^\circ = -3^\circ. \quad (53)$$

In this case³ the definition of correlation length as applied by Moritz (1980) is used. If we take the alternative definition of correlation length, which is the abscissa value of $1/e$ of the variance, we obtain

$$\delta \ell_e = 13.5^\circ - 17^\circ = -3.5^\circ. \quad (54)$$

The latter definition, which has been taken from mechanics, is customarily used in statistics.

The values in eqs. (53) and (54) are related to the original GEM 10 data. If the degree variances are referred to a sphere of radius R_E (table 1b, columns A' and B') we obtain analogously

$$\delta \ell = 10.5^\circ - 14.0^\circ = -3.5^\circ \quad (55)$$

³ Vaníček and Grafarend (1980) call this type of correlation length the "radius of statistical semi-independence," denoting it as "the distance at which the normalized covariance drops to 0.5."

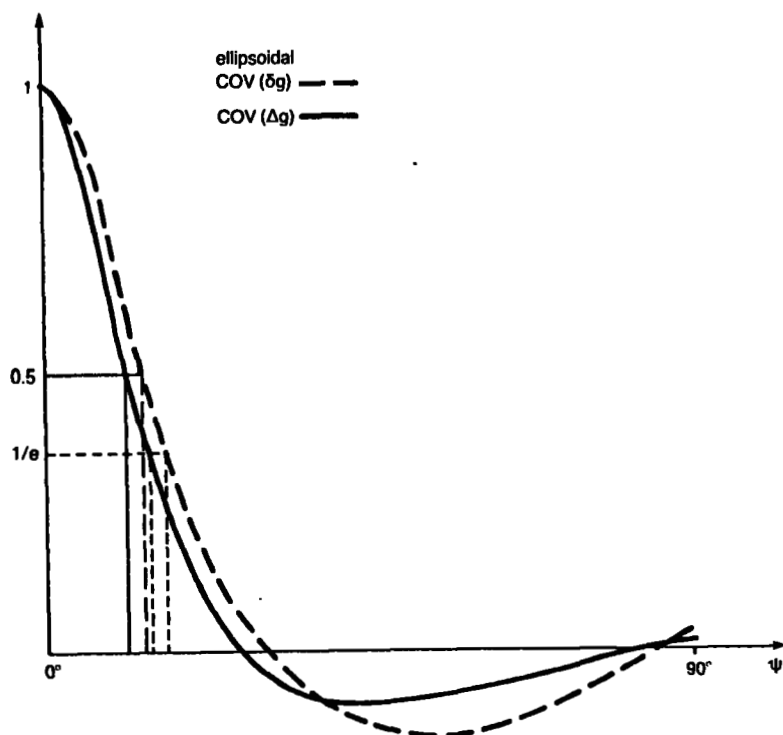


Figure 2.--Partial sum of degree variances related to GEM 10 data, showing the difference of correlation lengths $\delta\ell$ and $\delta\ell_e$, respectively.

and

$$\delta\ell_e = 14.0 - 17.0 = -3.0^\circ. \quad (56)$$

Figure 3 represents the curves analogous to the sums from (52) for Δg and σg related to the sphere.

Actual Gravity Investigation

Using realistic error covariances, Ihde (1980) has studied truncation errors for Stokes' and Vening-Meinesz's formulas. It should be pointed out that even though error covariances are supposed to be almost the same for δg and Δg the covariances of the function themselves are, of course, different because the covariances of Δg contain the term

$$\frac{\partial Y}{\partial h} N \approx 0.3086 N \quad (57)$$

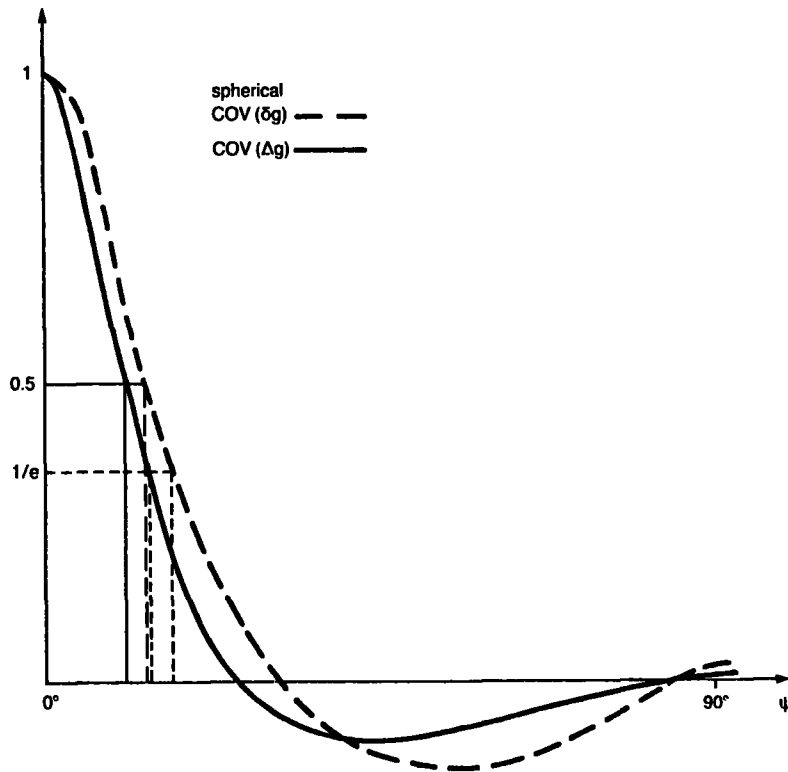


Figure 3.--Partial sums of degree variances related to a sphere of radius R_E , showing the difference of correlation length $\delta\ell$ and $\delta\ell_e$, respectively.

which is not found in the gravity disturbances. Thus, the covariance of $T=Ny$ has to be considered in addition to the covariance of the original function, Δg . Covariance propagation then gives the transition

$$\text{cov} (\delta g, \delta g) \rightarrow \text{cov} (\Delta g, \Delta g) \quad (58)$$

explained in geodetic textbooks, e.g., Moritz (1980). (For detailed numerical results related to degree variance models, see Becker (1980).) Next we shall confine the discussion to error covariances that are used to obtain additional information about truncation errors associated with the δg and Δg formulas, respectively. Formulas for geoid undulation and deflection of the plumb line will be considered.

The data presented in figures 2 and 3 can be interpreted in two ways. We can either consider the sums in table 1a and figure 2 as partial sums or as a band-limited part of the power spectrum. Moreover, we can consider them as the sums that correspond to autocovariance functions of an Earth model truncated at degree $n=30$, such that all harmonic coefficients for degrees $n>30$ vanish. In the latter case we have a non-negative or semi-positive definite function. To possess a unique inverse, \underline{C}^{-1} in eq. (5), \underline{C} must be regular. However, if some of the coefficients σ_n (the degree variances) in eq. (17) vanish, the corresponding autocovariance matrix \underline{C}

will be singular. Moritz (1980) has discussed how to avoid the inversion of C by building up C^{-1} . Consequently, in principle, at least, truncated Earth models can also be handled.

Test computations clearly reveal that the increase in correlation length associated with the transition from Δg to δg does not mean a stronger influence of remote zones in eq. (5). This increase is compensated by the behavior of $K(T, \delta g)$ in comparison to $K(T, \Delta g)$. (See again eq. (5).) The cross-covariance matrix K is found from covariance propagation. This fact agrees with the truncation error investigation using the GEM 10 field.

Error covariances are well known as functions of the autocovariances of the function itself. Since gravity is a nonstationary function at the Earth's surface, we cannot expect the same covariance function over the entire Earth. Moreover, surface gravity coverage of the Earth is quite irregular, especially at sea. Nevertheless, it makes sense to obtain some average information by using the following error variances for mean anomalies of blocks of different sizes (Ihde 1980).

Table 2 shows the variances associated with smooth topography. Those for flat areas are given in parentheses. In comparison to these error variances the results expected from the GRAVSAT mission should have an accuracy of ± 2 to ± 5 mgal (corresponding to variances of 5 to 21 mgal^2), which are impressive (Douglas et al. 1980). Moreover, Rapp (1979) reported an accuracy of about ± 7 mgal for 1° -by- 1° mean gravity values. The data were obtained at sea from GEOS-3 altimetry.

Table 2.--Variance of gravity anomalies vs. block size

Block size	Error variance adopted	
	(mgal^2)	Flat (mgal^2)
$5^\circ \times 5^\circ$	1311	
$1^\circ \times 1^\circ$	777	(150)
$30' \times 30'$	535	(75)
$6' \times 6'$	106	(10)
10 km x 10 km	89	(10)
5 km x 5 km	22	(2)
2 km x 2 km	1.4	(1)

Even higher accuracy is expected from SEASAT altimeter data. If we assume an accuracy of ± 5 mgal for SEASAT data, then this accuracy would compare favorably with the ± 28 mgal obtained from table 2 for 1° -by- 1° mean values. Therefore, in view of the forthcoming new results it makes sense to assume an accuracy of about ± 6 mgal or even better in all oceanic areas, i.e., for about 70 percent of the Earth's surface. Alternatively, an accuracy on the order of ± 4 mgal or better can presumably be used after 1986 if GRAVSAT has been launched. This latter estimate corresponds to the

5km-by-5km mean values given in table 2. This clearly shows the forthcoming progress as well as the progress achieved during the last few years. The error estimates in table 2 can be considered as basically uncorrelated.

Douglas et al. (1980) believe that higher accuracies are possible. Their general conclusion states: "Thus it seems apparent that a suitably optimized 1 year low-low satellite mission could produce mean anomalies at the 1°-by-1° level to 1-mgal precision."

A preliminary estimate of the maximum density of gravity stations in the nearest neighborhood of the computational point is again obtained by applying Kaula's (1966) rule-of-thumb, assuming its validity up to very high harmonics. Even though this has never been verified, the assumption might be useful for obtaining a preliminary estimate. Chovitz's (1973) formula deduced from Kaula's rule yields

$$\frac{64}{n} = 0.05 \text{ m} \quad (59)$$

to obtain the harmonic degree n for an expected 5-cm error truncation. Consequently, we obtain $n = 1280$. This corresponds to 0°14 or 8'4. For 3 cm we obtain analogously 5'1. Therefore, we have to use a spacing of about 3 and 4 km, respectively, in the nearest neighborhood of the station in order to account for harmonics of degree $n > 1280$ and $n > 2133$, respectively. In this spacing the station density for the downward continuation of gravity according to Bjerhammar's concept has not been included yet. Because the latter depends so strongly on the smoothness of the topography, it is difficult to give a generally valid estimate. On the other hand, to establish a dense net within a small cap around the computational point is not a serious problem.

Douglas et al. (1980) based their accuracy estimate of truncated geoid heights on

$$\frac{64}{n} \rightarrow \frac{64}{180} = 30 \text{ cm.} \quad (60)$$

This corresponds to neglecting harmonics beyond $n = 180$. However, it does not fully account for mean anomalies in the 1°-by-1° blocks that also contain higher harmonics of various degrees. A reasonable estimate of the truncation effect might be obtained by considering that Douglas et al. (1980) used discrete gravity values at 0°5 spacing to estimate mean values of 1° blocks. Therefore,

$$\frac{64}{n} \rightarrow \frac{64}{360} = 15 \cdot \text{cm} \quad (61)$$

could be more realistic, but such a change from 30 cm to 15 cm would not remarkably affect the aforementioned spacing in the neighborhood cap.

10. NUMERICAL INVESTIGATION

Let us assume that the term in eq. (57), i.e.,

$$\Delta g - \delta g = \frac{\partial \gamma}{\partial h} N,$$

can be determined without any significant error, because the geoid undulations are known to within an error of <2 m, which implies that errors in eq. (57) are <1 mgal. Hence δg and Δg are obtained to the same accuracy.

The advantage of using δg instead of Δg can then be readily tested for geoid undulations and plumb line deflections using the standard deviations

$$m_p^2(N) = \frac{R^2}{8\pi\gamma^2} \sum_i m_i^2 \overline{(\Delta g)} \Delta s_i \int_{o_i} S^2(\psi) \sin \psi \, d\psi \quad (62)$$

$$m_p^2(N) = \frac{R^2}{8\pi\gamma^2} \sum_i m_i^2 \overline{(\delta g)} \Delta s_i \int_{o_i} H^2(\psi) \sin \psi \, d\psi \quad (63)$$

where $S(\psi)$ = Stokes' function,

$H(\psi)$ = Hotine's function,

o_i denotes the i -th ring of width $(\psi_{i2} - \psi_{i1})$ around the point where N is computed,

Δs_i is the area of o_i , and

$m(\overline{\Delta g})$ and $m(\overline{\delta g})$ are the errors of the mean values of gravity anomaly and gravity disturbance, respectively, used in o_i .

If the ring o_i is replaced by compartments we obtain

$$m_P^2(N) = \frac{R^2}{16\pi^2\gamma^2} \sum_j S^2(\bar{\psi}) \Delta s_j^2 m_j^2(\bar{\Delta g}) \quad (64)$$

$$\underline{m}_P^2(N) = \frac{R^2}{16\pi\gamma^2} \sum_j H^2(\bar{\psi}) \Delta s_j^2 m_j^2(\bar{\delta g}) \quad (65)$$

where Δs_j now denotes the area of the j -th compartment at "mean" distance ψ from the point of calculation.

Utilizing the aforementioned assumption

$$m^2(\bar{\Delta g}) \doteq m^2(\bar{\delta g}) \quad (66)$$

we obtain the ratio

$$M = \left(m_P^2(N) / \underline{m}_P^2(N) \right)^{1/2} \quad (67)$$

To determine the error contribution caused by the zone outside a cap of radius ψ_0 (we may assume that within the cap δg and Δg are perfectly known) we can restrict the summation in eqs. (62) to (65) to $\psi > \psi_0$. Figure 4 shows the ratio for $0.05^\circ \leq \psi < 40^\circ$ in eq. (67).

When the same reasoning is applied for computing the deflections of the vertical we obtain the following formulas which are analogous to eqs. (66) and (67).

$$m_P^2(\xi, \eta) = \frac{1}{16\pi\gamma^2} \sum_i m_i^2(\bar{\Delta g}) \Delta s_i \int_{o_i} S'^2(\psi) \sin \psi \, d\psi \quad (68)$$

$$\underline{m}_P^2(\xi, \eta) = \frac{1}{16\pi\gamma^2} \sum_i m_i^2(\bar{\delta g}) \Delta s_i \int_{o_i} H'^2(\psi) \sin \psi \, d\psi \quad (69)$$

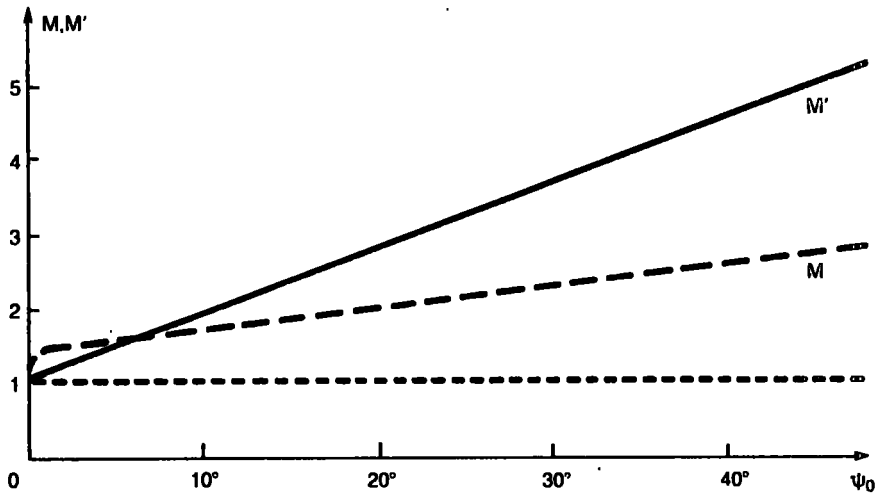


Figure 4--Error influence of remote zones outside of a cap of radius ψ_0 , where ψ_0 varies between the limits of $0^\circ 05'$ and $40^\circ 00'$.

$$m_P^2(\xi, \eta) = \frac{1}{32\pi^2\gamma^2} \sum_j S'^2(\bar{\psi}) \Delta s_j^2 m_j^2(\bar{\Delta}g) \quad (70)$$

or

$$\underline{m}_P^2(\xi, \eta) = \frac{1}{32\pi^2\gamma^2} \sum_j H'^2(\bar{\psi}) \Delta s_j^2 m_j^2(\bar{\delta}g) \quad (71)$$

with $S' \equiv dS/d\psi$ and $H' \equiv dH/d\psi$. Therefore, the ratio

$$M' = \left(m_P^2(\xi, \eta) / \underline{m}_P^2(\xi, \eta) \right)^{1/2} \quad (72)$$

which is analogous to eq. (67), can be introduced. Figure 4 shows the ratio for $0.05^\circ < \psi_0 < 40^\circ$ in eq. (72) by a broken line.

In an earlier study (Groten 1980), we showed that with the assumption given in eq. (66) the error influence of the remote zones $\psi > 50^\circ$ is smaller for Hotine-type integrals compared to Stokes' and Vening-Meinesz's integrals.

Since Stokes' equation reads

$$N(\underline{p}) = \frac{R}{4\pi\gamma} \iint_S \Delta g \, S ds = \frac{1}{4\pi\gamma R} \iint_{S'} \Delta g \, S ds'$$

where s is the unit sphere and s' is the terrestrial sphere of radius $r = R$, we can readily evaluate eqs. (62) and (64) by summing up over the unit sphere. We can likewise do the same for the Hotine integral.

Using accuracy estimates for 1° -by- 1° mean anomalies, as found by various authors, we obtain the results for geoidal height accuracies shown in table 3.

Table 3.--Accuracies for geoid height and deflection of the vertical to ± 2 , ± 4 , and ± 6 mgal for mean gravity in 1° -by- 1° blocks

Case	$m(\Delta g)$ (mgal)	$m(\delta g)$ (mgal)	$m(N)$ (cm)	$m(\xi, \eta)$
a)	± 1	--	± 8	$\pm 0''04$
	--	± 1	± 5.5	$\pm 0''03$
b)	± 2	--	± 16	$\pm 0''07$
	--	± 2	± 11	$\pm 0''06$
c)	± 4	--	± 32	$\pm 0''13$
	--	± 4	± 22	$\pm 0''13$
d)	± 6	--	± 48	$\pm 0''18$
	--	± 6	± 33	$\pm 0''20$

The values in table 3 were evaluated by assuming that perfect gravity information is available in a cap of 1° radius around a station. Therefore, these estimates are slightly optimistic. However, Groten and Moritz (1964) have shown how the errors within a cap of radius ψ_0 could be included. By applying this method, slightly higher values are found for $m(N)$ and $m(\xi, \eta)$. Moreover, the results in table 3 are too optimistic (as far as absolute geoid heights and deflections are concerned) for the following reasons: The distance between the GRAVSAT satellite pair is selected in such a way that maximum resolution is achieved for wavelengths having about 1° separations. This means that the low harmonics (as well as substantially higher harmonics) are not recovered to the same extent by GRAVSAT. Therefore, investigations of accuracy, such as the study by Douglas et al. (1980), tend to overestimate slightly the accuracy of GRAVSAT results. (The GRAVSAT system can be compared to a gravity gradiometer, which is sensitive to high harmonics, being tuned to a specific wavelength of the gravity field directly related to the separation of the satellite pair.) By accounting for accuracies on the order of ± 6 mgals or better for 1° -by- 1° mean gravity values, which were obtained from satellite altimetry, we may safely consider results of ± 2 to ± 4 mgal from GRAVSAT as realistic.

In addition, the results in table 3 show that the Hotine integral solution gives more favorable results. This is primarily important for deflections of the vertical whenever a cap of radius ψ_0 is covered by a dense terrestrial gravity field, as in the United States, where GRAVSAT data plus terrestrial data are combined in one solution.

As far as deflections of the vertical are concerned (which lose their previous importance) the superiority of the Hotine integral is quite remarkable for $\psi_0 \gg 5^\circ$, as shown by Groten. However, for values of $\psi_0 < 1^\circ$ the superiority is no longer as significant as for larger values of ψ_0 . This means that the contribution of the remote zones is prominent, whereas, when GRAVSAT-type data are available (i.e., with a good gravity field for harmonics of a high degree) this effect will still be as important as before. Contrary to this, in the present situation with only a small amount of reliable gravity material at hand, the superiority of the Hotine integral is dominant for the computation of the deflections of the vertical, as shown in figure 4. For example, figure 4 shows that the error budget of the remote zones for $\psi_0 > 20^\circ$ (which seems to be a realistic estimate for the United States and parts of western Europe) is almost three times larger when (ξ, η) is computed with the Vening-Meinesz method than computations made with the corresponding Hotine integral.

Since about two-thirds of the Earth are covered by the oceans and are satisfactorily surveyed by satellite altimetry, table 2 (case c) illustrates current achievable limits in terms of relative geoid heights. Because of the strong influence in the neighborhood zone, it is well known that the computation of (ξ, η) relies upon combinations of terrestrial and satellite data.

Therefore, a more detailed look at this problem is necessary. Consider the two cases: $\psi_0 = 6^\circ$ and $\psi_0 = 10^\circ$. The accuracies of (ξ, η) , corresponding to ± 1 mgal and ± 3 mgal, respectively, are shown in table 4. The results in

columns 4 and 5 pertain to a zero-error contribution from the neighborhood area of radius ψ_0 around the station where (ξ, η) is evaluated.

Table 4. -- Error contributions for computing deflections of the vertical using Vening-Meinesz's and Hotine's equations

1	2	3	4	5	6	7
					Total $m(\xi, \eta)$ for $\psi_0 = 6^\circ$ 10km x 10km 2km x 2km	
			$m(\xi, \eta)$		Flat to-	Regular to-
	$m(\Delta g)$	$m(\delta g)$	$\psi_0 = 6^\circ$	$\psi_0 = 10^\circ$	pography	pography
Case	(mgal)	(mgal)	(arc sec)	(arc sec)	(arc sec)	(arc sec)
a	± 2	--	± 0.01	± 0.01	± 0.23	± 0.02
	--	± 2	± 0.01	± 0.01	± 0.23	± 0.02
b	± 4	--	± 0.03	± 0.02	± 0.23	± 0.03
	--	± 4	± 0.02	± 0.01	± 0.23	± 0.02
c	± 6	--	± 0.05	± 0.03	± 0.24	± 0.04
	--	± 6	± 0.03	± 0.02	± 0.23	± 0.03

When we consider the error variances adopted at present for terrestrial data we obtain the following reasonable estimates: for a 2km- by-2km spacing (as in some European countries) up to a 6'-by-6' spacing (as in part of the Federal Republic of Germany), variances range between 100 mgal² and 1 mgal². When these error contributions are added to the results shown in table 4, columns 4 and 5, we obtain the results shown in columns 6 and 7. The innermost cap (nearest neighborhood of the station) of radius $\psi_0 = 5$ km is again considered as being error-free because this area is usually covered by an extremely dense local gravimetric survey.

It is realized that the superiority of the H' kernel over the S' kernel is relatively small in the area of $\psi \leq 5^\circ$. If we assume in table 4 that $m(\Delta g) < m(\delta g)$, we may compare $m(\Delta g)$ in case a with $m(\delta g)$ in cases b or c.

The effort necessary to obtain precise deflections of the vertical is readily seen in table 4. The strong influence of the station neighborhood even for the Hotine integral is shown in table 5, where

$$J' = \int_u^{180^\circ} (H')^2 \sin \psi \, d\psi$$

is listed for $0^\circ 05' \leq \psi \leq 10^\circ$. For comparison, the corresponding values for the integral

$$J = \int_n^{180^\circ} H^2 \sin \psi \, d\psi$$

are also listed.

Table 5.--Error integrals of the derivative of Hotine's functional derivative and of the function itself

u (degrees)	J'	J
10	48.0418	3.2136
6	150.4540	4.5638
5	223.4276	5.0318
4	360.1944	5.7703
3	660.7612	6.6893
1.5	2774.3723	9.0620
1	6347.7864	10.5281
0.9	7863.9416	10.9170
0.8	9982.8535	11.3550
0.7	13083.0679	11.8556
0.6	17868.7449	12.4384
0.5	25821.4937	13.1345
0.4	40474.9585	13.9822
0.3	72207.7578	15.0957
0.2	163011.7637	16.6720
0.1	654299.3984	19.3981
0.05	2621734.5937	22.1604

It seems appropriate to supplement the results in table 3 (where much terrestrial survey work is involved in order to fill up the innermost zone by a perfect gravity survey) by a less challenging effort where a station spacing of the terrestrial survey is supposed to be 2 km within a ring of $0^\circ 3'$ around the station. For $\psi < 0^\circ 05'$ the knowledge of the gravity field is supposed to be perfect. For $\psi > 0^\circ 3'$ we anticipate an accuracy of ± 1 mgal and ± 2 mgal, respectively. The results are shown in table 6.

Table 6.--Accuracies of geoid undulations

1	2	3	4	5	6
Case	$\psi > 0^{\circ}3$ (cm)	$0^{\circ}05$ Flat topo- graphy (cm)	$\leq \psi \leq 0^{\circ}3$ Regular topo- graphy (cm)	Total Flat topo- graphy (cm)	Regular topo- graphy (cm)
a	± 9.0	± 0.4	± 0.4	± 9.0	± 5.0
b	± 18.0	± 0.4	± 0.4	± 18.0	± 18.0
c	± 6.7	± 0.4	± 0.4	± 6.8	± 6.8
d	± 13.5	± 0.4	± 0.4	± 13.5	± 13.5

Case a denotes an accuracy of ± 1 mgal for $\psi > 0^{\circ}3$. Case b denotes an accuracy ± 2 mgal for $\psi > 0^{\circ}3$ in the Stokes approach. Cases c and d refer to the same accuracies using the Hotine integrals. In reality, geoid heights are, of course, directly obtained from the disturbing potential.

Table 7.--Accuracies of the deflections of the vertical

Case	$\psi > 0^{\circ}3$ (arc sec)	$0^{\circ}05$ Flat topo- graphy (arc sec)	$\leq \psi \leq 0^{\circ}3$ Regular topo- graphy (arc sec)	Total Flat topo- graphy (arc sec)	Regular topo- graphy (arc sec)
a	± 0.11	± 0.02	± 0.02	± 0.11	± 0.11
b	$\pm .22$	$\pm .02$	$\pm .02$	$\pm .22$	$\pm .22$
c	$\pm .11$	$\pm .02$	$\pm .02$	$\pm .11$	$\pm .11$
d	$\pm .22$	$\pm .02$	$\pm .02$	$\pm .22$	$\pm .22$

The accuracies in tables 6 and 7 refer to absolute geoid heights and deflections of the vertical. It is well known from the principle of astrogravimetric leveling that the remote zone effects are cancelled in determinations of relative geoid undulations. A conservative estimate is obtained when we assume that for station distances of $d \leq 100$ km the effect of the zones at a distance of $D \geq 10d \pm 10^{\circ}$ is cancelled. Using the Hotine integral for relative geoid undulations we obtain a reduction of 11 percent. By using the Stokes solution we obtain a reduction of 20 percent. Consequently, in table 6 we obtain an accuracy of about ± 7 cm instead of ± 9.0 cm and about ± 6 cm instead of 6.8 cm. (See columns 2 and 6.) The ± 18.0 -cm entry would decrease to ± 15 cm, and from ± 13.5 cm it would drop to ± 12 cm. These results reveal the importance of having an accuracy level in gravity on the order of ± 1 mgal or ± 2 mgal.

Consequently, by using GRAVSAT data with an accuracy of ± 1 or ± 2 mgal, supplemented by a local terrestrial gravity survey, we are able to determine relative geoid heights to an accuracy of ± 4 to ± 5 cm. Absolute geoid heights are found with slightly lower accuracy, on the order of ± 5 to ± 7 cm. However, atmospheric uncertainty and the zero-order uncertainty inherent in present GM values must be added. In the meantime (prior to the launch of GRAVSAT) we are obtaining relative geoid heights of slightly lower accuracy in well surveyed areas, i.e., around ± 10 cm, whereas absolute geoid heights are still affected by uncertainties of at least ± 50 cm to ± 60 cm. This is readily seen by replacing the gravity variances used in tables 3 to 7 with those of the aforementioned mean anomaly variances for 1° -by- 1° blocks.

The method discussed here is based on several hypotheses that were mainly discussed by Groten and Moritz (1964). It was applied to this study in a slightly different way. This should lead to only slightly pessimistic estimates for accuracies of N and (ξ, η) . Consequently, these estimates should be considered as comparative, in general, rather than absolute.

From the experience of dealing with downward continuation of Δg we may consider that an adequately surveyed cap of 10-km radius is sufficient for continuation of δg and Δg downward to a reasonable Bjerhammar sphere. Therefore, a neighborhood zone of radius $\psi_0 = 0.4$ with a dense gravity field is considered to be sufficient, in general, if the aim is to attain the results shown in tables 6 and 7.

A Remark on Series Representations

By using Kaula's rule of thumb, where the degree variances of the geopotential coefficients behave like $10^{-5}/n^2$, together with formula (31a) of Molodenskii et al. (1962, p. 166),

$$\frac{1}{\gamma^2} \sum_{n=2}^{\infty} Q_n^2 n(n+1) \sigma_n(\Delta g) = \sum_{n=2}^{\infty} Q_n^2 (n-1)^2 \sigma_n(\theta). \quad (73)$$

(The previous quantity $\overline{\theta^2}$ is now identified as the degree variance.) We realize that for $n \rightarrow \infty$

$$\sigma_n(\Delta g) = \frac{(n-1)^2 \gamma^2}{n(n+1)} \sigma_n(\theta) \rightarrow \sigma_n(\Delta g) / \sigma_n(\theta) = \text{constant}. \quad (74)$$

Moreover, by using $10^{-5}/n^2$, it is seen that, because of eq. (20), $\sigma_n(\theta)$ and $\delta_n(\Delta g)$ tend to be constant for larger values of n . In other words, neither the series for Δg nor σg nor (ξ, η) should converge under these assumptions.

Consequently, for small values of ψ_0 we cannot expect to obtain much information from eq. (31) and similar equations. The utilization of integrals of the type given in eq. (12) seems to be more efficient when we start from standard deviations of Δg or δg themselves.

On the other hand, this method was thoroughly studied by Jekeli (1979) and others, so that any verification using Δg_n and δg_n is no longer of interest. However, the application of the series approach to the single layer method, as shown in appendix 3, is of interest.

11. DISCUSSION

Several methods are available which, if properly applied, yield accuracies corresponding to the accuracy of results that may be expected from GRAVSAT and the Global Positioning System satellites, i.e., about ± 1 or ± 2 mgals and ± 5 cm in geoid height. If relative quantities such as geoid height differences are considered, then the correction for atmospheric uncertainties as well as the uncertainties inherent in the vertical datums and in GM (causing an error, δN_0) do not play a significant role. With GPS at hand, a very high accuracy of ± 1 cm or so is primarily expected for distances less than 100 km. In those cases the atmospheric effects could certainly be neglected almost everywhere and in almost all cases.

The comparison of various methods dealt mainly with δg versus Δg . One principal part of the investigation concerned the truncation error that related to various approximations and representations of the gravity field. Now the error covariance depends basically on the covariance of the function itself (Heiskanen and Moritz 1967, p. 268). Let the truncation error be represented as a product $a \cdot b$, where a is basically a function of the error variances and

$$b = \int_{\psi_0}^{\pi} k^2 \sin \psi \, d\psi.$$

It is realized that with kernel functions k the error of truncation e depends on the total power in Δg , δg , and μ rather than on specific parts of the spectrum. Consequently, it does not depend on a specific frequency band.

When Hotine's function H is compared with Stokes' function and when the corresponding two approaches

$$\begin{aligned} T &= \frac{R}{4\pi} \int \Delta g \, S \, ds = \frac{R}{4\pi} \int \delta g \, H \, ds \\ &= R \sum \frac{\Delta g_n}{n-1} = R \sum \frac{\delta g_n}{n+1} \end{aligned}$$

are compared with each other we apply the identity

$$\frac{(n-1)}{(n+1)} \cdot \frac{(n+1)}{(n-1)} = 1$$

where the first factor is applied to the observed gravity; whereas the second is applied to a kernel, such as H , which is independent of the data. In the case of collocation, the square of those factors must be considered.

As far as the downward continuation of μ , δg , and Δg is concerned we have discussed various aspects for obtaining μ , δg , and Δg at different levels. With δg and Δg the interior potential should be clearly distinguished from the analytical continuation of the external potential down into the terrestrial masses. Since

$$\lim_{r \rightarrow \infty} T(r) = 0 \quad \text{and} \quad \lim_{r \rightarrow \infty} N(r) = 0$$

we have, in general, increased separation between level surface $W = \text{constant}$ and the corresponding surfaces $U = \text{constant}$ with increasing depth below the Earth's surface for the analytical continuation of the external potential. Thus, N also increases. Consequently, the analytical continuation $\delta g = -\partial T / \partial r$ is assumed to be simpler, in general, than the continuation of Δg at greater depth.

As soon as GPS-type satellites are applied routinely for determining station location, gravity disturbances δg are expected to be fully equivalent to gravity anomalies Δg . If a satellite equipped with a high resolution sea altimeter is launched, δg will be more appropriate for geodetic purposes, in general, than Δg . With GRAVSAT data at hand, local and regional terrestrial data can be used for densifying satellite gravity in order to determine local geodetic parameters such as (ξ, η) whenever necessary.

When quantities pertaining to physical geodesy are used in combination with, or for comparison of, VLBI data, we mainly need relative geoid undulations. In that case, the uncertainties in GM do not play a significant role in the definition of a precise geoid. Consequently, as soon as better gravity data are available, an accuracy of ± 5 cm in relative geoid undulations will not be problematic. In the case of absolute values, however, it is still difficult to achieve an accuracy better than ± 30 cm, and it will be difficult to achieve an accuracy better than ± 20 cm even with better gravity data at hand for the reasons stated.

This study aimed at errors in gravity less than ± 0.006 mgal for each individual error source. It is a consequence of the relation $T_n = R \Delta g_n / (n-1) = R \delta g_n / (n+1)$ that the determination of relative geoid heights between stations not far apart from each other is the least problematic. Only part of the difficulties associated with a precise

definition of the geoid is dodged by using the quasi-geoid. However, because of the uncertainties inherent in the orthometric heights (caused by well-known hypotheses used in computing orthometric heights) it will be necessary to use normal heights and the quasi-geoid whenever ultimate accuracy is obtained. The basic accuracy limit of about 40 to 50 μgal seems to be reasonable in view of the corresponding accuracy in geoid heights or quasi-geoid heights obtained from

$$\delta N_n = \frac{R}{(n-1)\gamma} \delta (\Delta g).$$

For $n = 2$ we get

$$\delta N_n = 30 \text{ cm for } \delta (\Delta g) = 0.05 \text{ mgal}$$

by applying it to absolute geoid heights. Taking into account the north-south and east-west limits of the United States we have approximately $\Delta\lambda \doteq 60^\circ$ and $\Delta\phi \doteq 26^\circ$ which yields

$$\delta N = 5 \text{ cm}$$

for the same value of $\delta(\Delta g)$. Since the main source of uncertainty greater than 10 μgal in this study is the atmospheric correction within the United States, and the uncertainties could be assumed to be smaller than about $\pm 10 \mu\text{gal}$ (the assumption of an uncertainty of $\pm 50 \mu\text{gal}$ or $\pm 40 \mu\text{gal}$ in atmospheric correction is generally related to global phenomena), we could even assume

$$\delta N_n = \pm 1 \text{ cm}$$

corresponding to $\delta(\Delta g)$ of $\pm 10 \mu\text{gal}$. On the other hand, we also have to account for the other error sources in this case. Thus $\pm 3 \text{ cm}$ seems to be the highest accuracy achievable in relative determination of geoid height for the present "theory."

In a number of publications it has been shown that if the same hypotheses are used in the evaluation of orthometric and geoid heights the errors inherent in the hypotheses cancel (to a first-order approximation) each other. In reality, h and N are often taken from different sources. Therefore, in most cases it is safer to use height anomalies ζ and normal heights where the well-known relation applies

$$N - \zeta = -(\bar{g}/\bar{\gamma} - 1)h$$

where \bar{g} and $\bar{\gamma}$ are mean values of g and γ , which are taken along the plumb lines between the Earth's surface (geoid) and telluroid (ellipsoid), respectively.

Thus, if GRAVSAT provides the disturbing potential, yielding an accuracy of ± 30 cm or better in geoid height based on harmonics up to degree $n=180$, we can solve for higher harmonics of geoid heights using local gravity with an accuracy of about ± 5 cm.

12. CONCLUSION

Theoretically, it has been shown that we could obtain accuracies of about ± 3 to ± 4 cm for relative geoid heights. At higher accuracies, uncertainties prevail that cannot be solved even if a ± 1 mgal gravity field becomes available from GRAVSAT. For GRAVSAT data (with an accuracy of ± 1 mgal or ± 2 mgal) we can obtain, in combination with local gravity surveys (within the neighboring zone), relative geoid heights over distances of ± 100 km or so with an accuracy of ± 5 cm. Also deflections of the vertical that fulfill present requirements, i.e., with an accuracy of better than $\pm 0''3$, can be obtained. Using Hotine's integrals, the results are slightly better than those obtained with Stokes' integrals.

13. ACKNOWLEDGMENT

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The first derivative of eq. (2) yields

$$\frac{\partial H(r,\psi)}{\partial \psi} = 2k \frac{\partial \phi^{-1/2}}{\partial \psi} - \frac{\partial}{\partial \psi} \left[\ln \left(\frac{\sqrt{\phi} + k - \cos \psi}{1 - \cos \psi} \right) \right] \quad (A1)$$

with $k = R/r$. Further,

$$\frac{\partial H(r,\psi)}{\partial \psi} = -\sin \psi \left[\frac{2k^2}{\phi^{3/2}} - \frac{1}{1 - \cos \psi} + \frac{k \phi^{-1/2} + 1}{\sqrt{\phi} + k - \cos \psi} \right] \quad (A2)$$

For $k = 1$ we obtain

$$\phi = 4 \sin^2 (\psi/2). \quad (A3)$$

Moreover,

$$2 \sin^2 (\psi/2) = 1 - \cos \psi \quad (A4)$$

and

$$\sin \psi = 2 \sin \left(\frac{\psi}{2} \right) \cos \left(\frac{\psi}{2} \right). \quad (A5)$$

Consequently,

$$\frac{dH(\psi)}{d\psi} = \frac{\partial H(\psi)}{\partial \psi} = -\cos \left(\frac{\psi}{2} \right) \left[\frac{1}{2 \sin^2 \left(\frac{\psi}{2} \right)} - \frac{1}{\sin \left(\frac{\psi}{2} \right)} + \frac{2 \sin \left(\frac{\psi}{2} \right) + 1}{2 \sin^2 \left(\frac{\psi}{2} \right) + 2 \sin \left(\frac{\psi}{2} \right)} \right] \quad (A6)$$

which after a simple transformation yields (Groten 1979)

$$\frac{dH(\psi)}{d\psi} = -\frac{\cos \left(\frac{\psi}{2} \right)}{2 \sin^2 \left(\frac{\psi}{2} \right)} + \frac{\cos \left(\frac{\psi}{2} \right)}{2 \sin^2 \left(\frac{\psi}{2} \right) + 2 \sin \left(\frac{\psi}{2} \right)}. \quad (A7)$$

The second derivative of eq. (2) is readily derived from the first derivative using

$$\frac{\partial^2 H(r, \psi)}{\partial \psi^2} = \frac{\partial H_1}{\partial \psi} + \frac{\partial H_{12}}{\partial \psi} + \frac{\partial H_{22}}{\partial \psi} \quad (A8)$$

with
$$\frac{\partial H_1}{\partial \psi} = \frac{\partial}{\partial \psi} (-2k^2 \phi^{-3/2} \sin \psi) \quad (A9)$$

or
$$\frac{\partial H_1}{\partial \psi} = + 2k^2 \left[\frac{3}{2} \phi^{-5/2} \frac{\partial \phi}{\partial \psi} \sin \psi - \cos \psi \phi^{-3/2} \right] \quad (A10)$$

$$\frac{\partial H_1}{\partial \psi} = + 2k^2 \phi^{-3/2} \left[3k\phi^{-1} \sin^2 \psi - \cos \psi \right] \quad (A11)$$

$$\begin{aligned} \frac{\partial H_{12}}{\partial \psi} &= \frac{\partial}{\partial \psi} \left[- \left(k\phi^{-1/2} \sin \psi + \sin \psi \right) \left(\phi^{1/2} + k - \cos \psi \right)^{-1} \right] \\ &= \frac{k^2 \phi^{-3/2} \sin^2 \psi - k\phi^{-1/2} \cos \psi - \cos \psi}{\phi^{1/2} + k - \cos \psi} \\ &\quad + \frac{\sin^2 \psi (k\phi^{-1/2} + 1)^2}{(\phi^{1/2} + k - \cos \psi)^2}, \quad \text{and} \end{aligned} \quad (A12)$$

$$\begin{aligned} \frac{\partial H_{22}}{\partial \psi} &= \frac{\partial}{\partial \psi} \left(\frac{\sin \psi}{1 - \cos \psi} \right) \\ &= \frac{\cos \psi (1 - \cos \psi) - \sin^2 \psi}{(1 - \cos \psi)^2} \\ &= \frac{\cos \psi}{(1 - \cos \psi)} - \frac{\sin^2 \psi}{(1 - \cos \psi)^2}. \end{aligned} \quad (A13)$$

Consequently, we finally obtain from eq. (A8)

$$\begin{aligned}
\frac{\partial^2 H}{\partial \psi^2} = & 2k^2 \phi^{-3/2} \left(3k\phi^{-1} \sin^2 \psi - \cos \psi \right) \\
& + \frac{k^2 \phi^{-3/2} \sin^2 \psi - k\phi^{-1/2} \cos \psi - \cos \psi}{\phi^{1/2} + k - \cos \psi} + \left(\frac{k\phi^{-1/2} \sin \psi + \sin \psi}{\phi^{1/2} + k - \cos \psi} \right)^2 \\
& + \frac{\cos \psi}{(1 - \cos \psi)} - \frac{\sin^2 \psi}{(1 - \cos \psi)^2}.
\end{aligned} \tag{A14}$$

For $k = 1$ we end up with the corresponding formula of Groten's (1979), i.e.,

$$\frac{d^2 H}{d\psi^2} = \frac{1}{2\sin^3(\psi/2)} - \frac{1}{4\sin \psi/2} + \frac{\sin^3(\psi/2) - 2\sin(\psi/2) - 1}{4[\sin^2(\psi/2) + \sin(\psi/2)]^2} \tag{A15}$$

$$\text{using } \frac{\partial^2 H}{\partial \psi^2} = \text{I} + \text{II} + \text{III} \tag{A16}$$

$$\text{where } \text{I} = \frac{-2\cos \psi}{\phi^{3/2}} + \frac{6\sin^2 \psi}{\phi^{5/2}}, \tag{A17}$$

$$\begin{aligned}
\text{II} = & - \frac{\cos \psi + \cos \psi \phi^{-1/2} - \phi^{-3/2} \sin^2 \psi}{1 - \cos \psi + \phi^{1/2}} \\
& + \left(\frac{\sin \psi + \phi^{-1/2} \sin \psi}{1 - \cos \psi + \phi^{1/2}} \right)^2.
\end{aligned} \tag{A18}$$

$$\text{III} = \frac{2 \cos \psi}{\phi} - \frac{4 \sin^2 \psi}{\phi^2}. \tag{A19}$$

Now straightforward algebraic manipulation leads to

$$\begin{aligned}
 I &= \frac{-2\cos \psi}{8\sin^3(\psi/2)} + \frac{6\sin^2 \psi}{32\sin^5(\psi/2)} \\
 &= \frac{-1}{4\sin^3(\psi/2)} + \frac{1}{2\sin(\psi/2)} + \frac{3}{4\sin^3(\psi/2)} - \frac{3}{4\sin(\psi/2)} \\
 &= \frac{1}{2\sin^3(\psi/2)} - \frac{1}{4\sin(\psi/2)} \quad (A20)
 \end{aligned}$$

$$\begin{aligned}
 III &= \frac{2\cos \psi}{4\sin^2(\psi/2)} - \frac{4\sin^2 \psi}{16\sin^4(\psi/2)} = \\
 &= \frac{1}{2\sin^2(\psi/2)} - 1 - \frac{1}{\sin^2(\psi/2)} + 1 = -\frac{1}{2\sin^2(\psi/2)} ; \quad (A21)
 \end{aligned}$$

$$\text{and } I + III = \frac{1}{2\sin^3(\psi/2)} - \frac{1}{4\sin(\psi/2)} - \frac{1}{2\sin^2(\psi/2)} ; \quad (A22)$$

$$II = -\frac{\cos \psi + \frac{\cos \psi}{2\sin(\psi/2)} - \frac{\sin^2 \psi}{8\sin^3(\psi/2)}}{2\sin^2(\psi/2) + 2\sin(\psi/2)} + \frac{\left[\sin \psi + \frac{\sin \psi}{2\sin(\psi/2)} \right]^2}{[2\sin^2(\psi/2) + 2\sin(\psi/2)]^2} \quad (A23)$$

$$II = \frac{-\cos \psi - \frac{\cos^2(\psi/2)}{2\sin(\psi/2)} + \frac{\sin(\psi/2)}{2} + \frac{\cos^2(\psi/2)}{2\sin(\psi/2)}}{2\sin^2(\psi/2) + 2\sin(\psi/2)} + \frac{(\sin \psi + \cos(\psi/2))^2}{4[\sin^2(\psi/2) + \sin(\psi/2)]^2} \quad (A24)$$

Using the abbreviations

$$x_1 = -2\sin^2(\psi/2)\cos \psi = -2\sin^2(\psi/2)\cos^2(\psi/2) + 2\sin^4(\psi/2)$$

$$x_2 = -2\sin(\psi/2)\cos \psi = -2\sin(\psi/2)\cos^2(\psi/2) + 2\sin^3(\psi/2)$$

$$x_3 = \sin^2 \psi = 4\sin^2(\psi/2)\cos^2(\psi/2)$$

$$x_4 = 2\sin \psi \cos(\psi/2) = 4\sin(\psi/2)\cos^2(\psi/2)$$

we obtain

$$II = \frac{\sum_{i=1}^4 x_i + \sin^3(\psi/2) + 1}{4 [\sin^2(\psi/2) + \sin(\psi/2)]^2} \quad (A25)$$

whence from eq. (A16) we obtain

$$\frac{\partial^2 H}{\partial \psi^2} = \frac{1}{2\sin^3(\psi/2)} - \frac{1}{4\sin(\psi/2)} - \frac{1}{2\sin^2(\psi/2)} + \frac{2\sin^2(\psi/2) + 2\sin(\psi/2) + 1 + \sin^3(\psi/2)}{4[\sin^2(\psi/2) + \sin(\psi/2)]^2}$$

or

$$\frac{\partial^2 H}{\partial \psi^2} = \frac{1}{2\sin^3(\psi/2)} - \frac{1}{4\sin(\psi/2)} +$$

$$\frac{2\sin^2(\psi/2) + 2\sin(\psi/2) + 1 + \sin^3(\psi/2) - 2\sin^2(\psi/2) - 4\sin(\psi/2) - 2}{4[\sin^2(\psi/2) + \sin(\psi/2)]^2} \quad (A26)$$

which is identical to eq. (10) taken from Groten (1979). Consequently, the derivations are correct.

APPENDIX B.--DISCUSSION OF DENSITY MODEL

Starting from the well-known equation

$$T(P) = 4\pi GR \sum_{n=0}^{\infty} \sigma_n \left(\frac{R}{r}\right)^{n+1} \frac{1}{2n+1} \quad (A27)$$

where G is the Newtonian gravitational constant. We obtain the disturbing potential at $P(r)$ caused by a single layer density on a sphere of radius $r=R$ having a density of

$$\sigma = \sum \sigma_n. \quad (A28)$$

This spherical harmonics representation is analogous to eq. (43). Differentiation then leads to (remembering the discontinuity of derivatives for $R=r$):

$$\left(\frac{\partial T}{\partial r}\right)_P = -4\pi G \sum \frac{n+1}{2n+1} \sigma_n. \quad (A29)$$

Or using

$$-\frac{\partial T}{\partial r} = \delta g = \sum \delta g_n \quad (A30)$$

we obtain

$$\delta g(P) = 4\pi G \sum \frac{n+1}{2n+1} \sigma_n. \quad (A31)$$

The Neumannian problem is solved by Heiskanen and Moritz (1967, p. 36)

$$T(P) = R \sum \left(\frac{R}{r}\right)^{n+1} \frac{\delta g_n}{n+1}.$$

For $r=R$ we obtain from eq. (20)

$$T_n(R) = \frac{R\delta g_n}{n+1}$$

and, analogously,

$$T_n(R) = \frac{R\Delta g_n}{n-1}$$

Using eqs. (25) and (38) we end up with

$$\mu_n = \left(n + \frac{1}{2}\right) \Delta g_n / (n-1) \quad (A32)$$

$$\mu_n = \left(n + \frac{1}{2}\right) T_n / R \quad (A33)$$

$$\mu_n = \left(n + \frac{1}{2}\right) \delta g_n / (n+1). \quad (A34)$$

With $n \rightarrow \infty$ we obtain

$$\left(n + \frac{1}{2}\right) / \left(n + 1\right) \doteq \left(n + \frac{1}{2}\right) / \left(n - 1\right) \rightarrow 1.$$

Consequently, for high values of n there is no basic difference between

δg_n , Δg_n and μ_n . For lower n , μ_n could be treated as a high-pass filtered δg_n , whereas Δg_n is basically considered as a high-pass filtered version of μ_n .

$$f = R / \left(n + \frac{1}{2}\right) \quad (A35)$$

can be considered as a low-pass filter function to be applied to μ_n in order to obtain T_n . The filter functions are illustrated in figure 5. They reveal the deviations from 1 for $n \leq 30$. Formulas (35) and (38) take into account the well-known discontinuity of single layer potentials in external space. (For more details see Orlin (1959).)

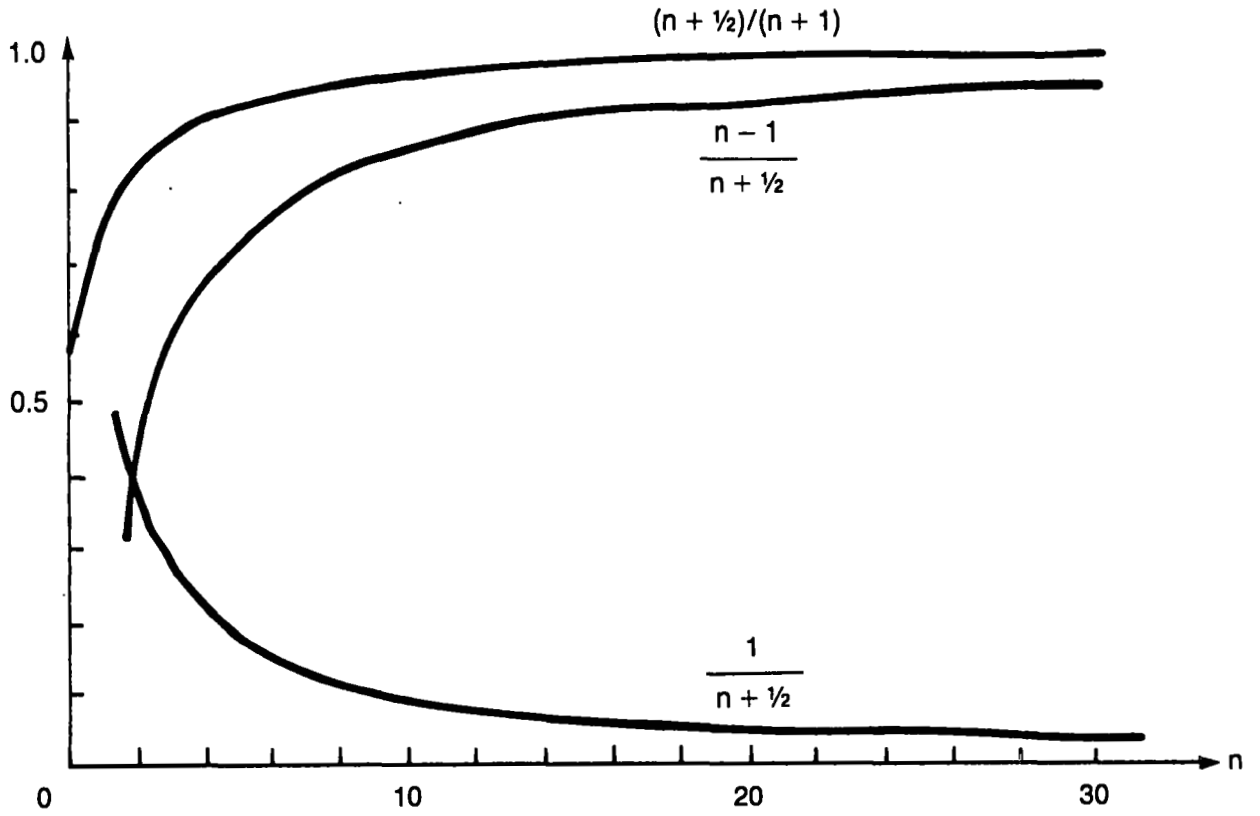


Figure 5.--Filter functions involved in σ_n representations.

From these results we conclude that the use of μ is nearly equivalent to the application of δg . The spherical approximation utilized above can be replaced by a rigorous application of a single layer density on an ellipsoid, or the sphere of radius $r = R$ can be considered as a Bjerhammar sphere, as defined in Moritz (1980, p. 69). The latter case then represents a rigorous solution in terms of the Runge-Krarup theorem (Moritz 1980, p. 64).

Moreover, we can split off the first two harmonics in eq. (43), leading to its spherical harmonic representation

$$T(P) = 4\pi GR \sum_{n=2}^{\infty} \frac{\sigma_n}{2n+1} \left(\frac{R}{r}\right)^{n+1}. \quad (A36)$$

Because, in eq. (43),

$$\frac{1}{\ell} = R^{-1} \sum_0^{\infty} \left(\frac{R}{r}\right)^{n+1} P_n(\cos\psi) \quad (A37)$$

we get for $r=R$ with $\varphi=2R \sin (\psi/2)$

$$\begin{aligned} \frac{1}{\ell} &= R^{-1} \left(\sum_2^{\infty} P_n (\cos\psi) + P_0 (\cos\psi) + P_1 (\cos\psi) \right) \\ &= R^{-1} \left(\sum_2^{\infty} P_n (\cos\psi) + 1 + \cos\psi \right) \end{aligned} \quad (A38)$$

or

$$\frac{1}{\ell'} = R^{-1} \sum_2^{\infty} P_n (\cos\psi) = \frac{1}{\ell} - R^{-1}(1+\cos \psi). \quad (A39)$$

The term $(1 + \cos\psi)$ is positive for all values of ψ . By subtracting it from $1/\ell$ we obtain a function $1/\ell'$.

Moreover, from the spherical harmonics representation of (43) we get the two terms missing in its last form, i.e.,

$$\sum_{n=0}^1 \frac{P_n (\cos\psi)}{2n+1} = 1 + \frac{1}{3} \cos\psi \quad (A40)$$

which again is positive definite. By subtracting the two first harmonics from the original form $1/\ell$, as shown in eq. (A39), we obtain a kernel function which corresponds to the modified "Hotine function" (Groten 1979) or Stokes' function.

It is interesting to see that even if σ_n ($n = 0,1$) is nonvanishing the potential T will have $T_0 = T_1 = 0$. Consequently the reasoning usually applied to Stokes' function can now be applied to the single layer integral formula. We use

$$\cos \psi = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos (\lambda-\lambda') \quad (A41)$$

for the point $P(\theta,\lambda)$ and the surface element ds (θ',λ') with formulas analogous to those found for Stokes' function in Heiskanen and Moritz (1967, p. 99). By inspecting $1/\ell'$ it is realized that this function deviates from $1/\ell$

mainly for small values of ψ . When we refer μ_n to the same sphere to which Δg_n is related we can compute the corresponding degree variances of μ_n related to the GEM 10 model. These are listed in table 8.

A comparison of degree variances of μ_n and δg_n shows a close connection as a consequence of eq. (A34). The reason we are in favor of handling the term in a classical Neumannian form lies in the fact that the uniqueness and existence of this boundary value problem have been well investigated. The problems of downward continuation of quantities down to the Bjerhammar sphere associated with smoothing and, consequently, with nonuniqueness do exist.

The question of whether the upward continuation up to a Brillouin sphere is preferred and whether the complete field itself could then be continued downward from a Brillouin sphere to a Bjerhammar sphere cannot be answered in this report. It certainly has advantages, but there have been too few numerical investigations. The really valuable investigations concerning the Brillouin approach were of a theoretical nature. Jeffreys' argument, "when in doubt, smooth" (Moritz 1980, p. 270), is certainly correct, but smoothed data are no longer precise data! Thus, upward and downward continuations applied together involve double smoothing.

Table 8.--Degree variances of μ and δg related to a sphere of mean Earth radius

n	μ_n (mgal)	δg_n (mgal)	n	μ_n (mgal)	δg_n (mgal)
3	103.8	135.6	17	2.6	2.8
4	44.8	55.3	18	3.8	4.0
5	39.5	47.0	19	2.5	2.6
6	32.6	37.8	20	2.4	2.6
7	30.5	34.7	21	2.2	2.3
8	17.3	19.3	22	2.1	2.2
9	16.1	17.8	23	0.6	0.6
10	13.6	14.9	24	0.5	0.5
11	9.0	9.8	25	0.8	0.8
12	4.8	5.2	26	0.0	0.0
13	8.1	8.7	27	0.7	0.7
14	4.4	4.7	28	1.1	1.2
15	3.8	4.0	29	0.7	0.7
16	3.3	3.5	30	1.4	1.5

As far as the Bjerhammar sphere is concerned, it has been defined by various authors (Moritz 1980, pp. 69, 181). When the radius is varied we obtain $\mu_n(R_1)$ from $\mu_n(R_2)$ and vice versa. Using eq. (A27) it immediately yields

$$\mu_n(R_1) \left(\frac{R_1}{R_2} \right)^{n+1} = \mu_n(R_2). \quad (\text{A42})$$

When the analogous reasoning is applied to δg_n or Δg_n we have $(n+2)$ in the exponent instead of $(n+1)$. This follows from differentiating the formula which solves the Dirichlet boundary value problem.

If the spherical treatment discussed above is considered as an approximation of the ellipsoidal problem then, of course, ellipsoidal corrections in their conventional form (Moritz 1980, p. 327) have to be applied.

When the same problems are considered in terms of collocation we may start from an autocovariance model of the disturbing potential. By writing eq. (A27) in symbolic form

$$T = I\mu \quad (\text{43a})$$

with the operator

$$I = \frac{R^2}{2\pi} \iint \ell^{-1} ds \quad (\text{A43})$$

being applied to $\mu(R)$ we have a unique relation between μ and T in contrast to the general problem (potential \leftrightarrow density) where (potential \rightarrow density) is well known to be ambiguous, in general. Without going into details we can obtain

$$\mu = I^{-1}T \quad (A44)$$

where I^{-1} is the inverse of I , the autocovariance being immediately obtained by using, e.g., eq. (A33), with $\sigma_n(\mu)$, the degree variances of "density" μ , being

$$\sigma_n(\mu) = \frac{(n+\frac{1}{2})^2}{R^2} \sigma_n(T). \quad (A45)$$

Analogous covariance propagation can be used for transitions $\mu \rightarrow \Delta g$, $\mu \rightarrow \delta g$, etc. The cross-covariances \underline{K} are, in general, found from

$$\underline{K} = I \underline{C} \quad (A46)$$

where I is the operator relating two functions, such as T and μ , \underline{C} being the autocovariance matrix of the latter function. The appropriate frame work in terms of stochastic processes on a sphere is given in Grafarend (1976).

Hotine (1969, p. 346) used the relation

$$\frac{(2n+1)^2}{n-1} = 4n+2+3 \frac{(2n+1)}{n-1}$$

in order to transform a formula almost identical with our relation (A32) into an integral formula, ending up with a relation which, using our notation, reads

$$\mu = \frac{1}{8\pi} \iint_S \Delta g Q(\psi) ds$$

or (A47)

$$\mu = \frac{1}{8\pi} \iint \Delta g Q(\psi) \sin \psi \, d\psi \, d\alpha$$

when ds is again an element of the unit sphere s , α = azimuth, and ψ is the spherical distance; moreover,

$$Q(\psi) = 1 + 3\operatorname{cosec}\left(\frac{\psi}{2}\right) - 18\sin\left(\frac{\psi}{2}\right) - 21\cos\psi - 9\cos\psi \ln\left[\sin\left(\frac{\psi}{2}\right) + \sin^2\left(\frac{\psi}{2}\right)\right]. \quad (\text{A48})$$

The inverse of this equation, for $\underline{r = R}$, is

$$\Delta g(P) = 2\pi G\sigma(P) - \frac{3}{4}G \iint_S \operatorname{cosec}\left(\frac{\psi}{2}\right) \sigma \, ds$$

with $\sigma = \frac{\mu}{2\pi} G$. (See Hotine 1969, p. 397.)

By inspecting the function $Q(\psi)$ it is readily seen that it behaves quite similar to Stokes' function for $\psi=0$ and $\psi=180^\circ$; it vanishes at $\psi=0$, and $Q(\psi=\pi)=13.2$. It is tempting to derive a similar relation connecting δg and μ . The corresponding function $\bar{Q}(\psi)$ in

$$\mu = \frac{1}{8\pi} \iint_S \delta g \, \bar{Q}(\psi) \, ds \quad (\text{A49})$$

is supposed to be much simpler than $Q(\psi)$ because of the relation in eq. (A34). Using the identity

$$\frac{2n+1}{n+1} = 2 - \frac{1}{n+1} \quad (\text{A50})$$

$$\frac{(2n+1)^2}{n+1} = 2(2n+1) - \frac{2n+1}{n+1}$$

we finally obtain

$$-\bar{Q}(\psi) = 1 + \frac{9}{2} \cos \psi + \operatorname{cosec}\left(\frac{\psi}{2}\right) - \ln \left[1 + \operatorname{cosec}\left(\frac{\psi}{2}\right) \right] \quad (\text{A51.1})$$

when we start with $n=2$ in the summation of the spherical harmonic representation of μ and δg , and

$$-\bar{Q}'(\psi) = \operatorname{cosec}\left(\frac{\psi}{2}\right) - \ln \left[1 + \operatorname{cosec}\left(\frac{\psi}{2}\right) \right] \quad (\text{A51.2})$$

when we start with $n=0$, respectively.

The intermediate steps in deriving the last formula are briefly summarized as follows:

Differentiate the well-known relation

$$(1 - 2k \cos \psi + k^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} k^n P_n(\cos \psi)$$

with respect to k , multiply the result by $2k$, and add it to the original equation, thus obtaining

$$\frac{1 - k^2}{(1 - 2k \cos \psi + k^2)^{3/2}} = \sum_{n=0}^{\infty} (2n+1) k^n P_n(\cos \psi)$$

or

$$\sum_{n=2}^{\infty} (2n+1) k^{n-2} P_n(\cos \psi) = -\frac{1}{k^2} \left[1 + 3k \cos \psi - \frac{1 - k^2}{(1 - 2k \cos \psi + k^2)^{3/2}} \right] \quad (\text{A52})$$

where $k = \frac{R}{r}$ which we set equal to 1.

From Hotine (1969, p. 311) we know that

$$\begin{aligned} H(\psi) &= \sum_{n=0}^{\infty} \frac{2n+1}{n+1} P_n(\cos \psi) \\ &= \operatorname{cosec}\left(\frac{\psi}{2}\right) - \ln \left[1 + \operatorname{cosec}\left(\frac{\psi}{2}\right) \right]. \end{aligned}$$

By using the latter equation and eq. (A52), we obtain from equation (A50) the relation (A51.1) and (A51.2), noting that

$$\bar{Q}(\psi) = \sum_{n=2}^{\infty} \frac{(2n+1)^2}{n+1} P_n(\cos \psi)$$

$$\bar{Q}'(\psi) = \sum_{n=0}^{\infty} \frac{(2n+1)^2}{n+1} P_n(\cos \psi)$$

Again, the inverse transformation is given by the well-known formula, which is found in slightly different form, for example, in Heiskanen and Moritz (1967, p. 236)

$$\delta g(P) = \mu(P) + \frac{R}{4\pi} \iint \mu \ell^{-1} ds$$

ℓ again being the distance \overline{Pds} , where P is located on s .

Equation (A49) enables us to express gravity disturbances within a local area or beyond by a single layer density. Such a layer can be directly compared to the corresponding topography.

A graphical representation of the function $\bar{Q}(\psi)$ of eq. (A51.1) is shown in figure 6. It should be noted that in eq. (A51.1) we used $-\bar{Q}(\psi)$ instead of $\bar{Q}(\psi)$. This results from defining gravity disturbances by $-\delta g = \partial T / \partial h$, whereas Hotine in his original studies introduced the position vertical gradient of the disturbing potential. The easiest way to make his results comparable to ours is to introduce a negative kernel function, as in eq. (A51.1). Figure 6 shows $\bar{Q}(\psi)$ as well as $\bar{Q}(\psi)\sin\psi$. Rigorously, formula (A49) corrects δg and μ if both are given on a Bjerhammar sphere. However, as far as the approximation

$$\frac{\partial g}{\partial h}(P) = \frac{\partial \gamma}{\partial h}(P)$$

holds, δg does not depend on the elevation. The situation is slightly simpler

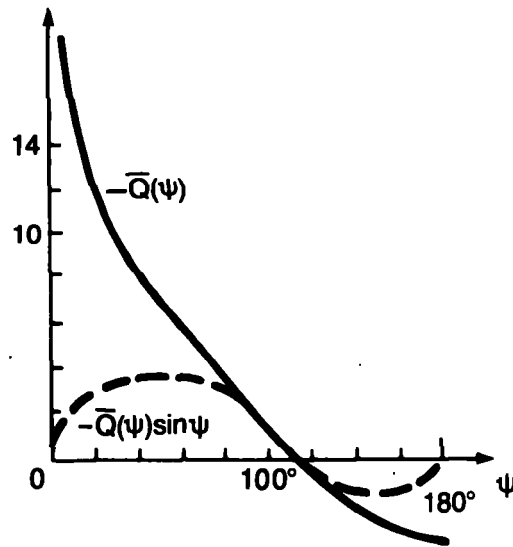


Figure 6. $-\bar{Q}(\psi)$ and $\bar{Q}(\psi)\sin\psi$.

than in the case of Δg where the corresponding relation reads

$$\frac{\partial g}{\partial h}(P) = \frac{\partial \gamma}{\partial h}(Q),$$

with P being on point on the geop and Q being a point on the spherop.

For interpolation of gravity using the cross-correlation with topography the above relations $\mu(\delta g)$ are of great practical impact. Since interpolation is always involved in precise work of physical geodesy the foregoing formulas are of importance to the problem discussed in this report. Instead of the aforementioned inverse of eq. (A49) we can, of course, also consider (A49) as an integral equation which is solved for the unknown δg using numerical methods, i.e., by replacing the integral by a summation process.

When we apply the same reasoning to μ_n , which Molodensky et. al. (1962, p. 147) applied to Δg_n and Groten and Jochemczyk (1978) applied to δg_n , we end up with

$$T = \frac{R^2}{2\pi} \iint_s \frac{\mu}{\ell} ds = \frac{1}{2\pi} \iint_S \frac{\mu}{\ell} dS$$

where again s and S denote the unit sphere and a sphere of radius $r=R$, respectively. On the sphere itself the kernel \bar{k} reads

$$\bar{k} = \ell^{-1} = [2\sin(\frac{\psi}{2})]^{-1}.$$

Care is again necessary when derivatives are considered for $r=R$ because of the discontinuity of derivatives on s and S , respectively, so that, e.g.,

$$\frac{\partial T}{\partial n} = \mu - \frac{R}{4\pi} \iint \frac{\mu}{\ell} ds$$

where ψ = center angle and n denotes the outer surface normal to s and S . By expanding the formula

$$\bar{k}(\psi) = \begin{cases} 0 & \text{if } 0 \leq \psi < \psi_0 \\ \ell^{-1} & \text{if } \psi_0 \leq \psi \leq \pi \end{cases}$$

we obtain

$$\bar{k}(\psi) = \sum \frac{2n+1}{2} k_n P_n(\cos\psi)$$

with

$$\begin{aligned} k_n &= \int_0^\pi \bar{k}(\psi) P_n(\cos\psi) \sin \psi \, d\psi \\ &= \int_{\psi_0}^\pi [2 \sin(\frac{\psi}{2})]^{-1} P_n(\cos\psi) \sin \psi \, d\psi \end{aligned}$$

and finally

$$\begin{aligned}
 T &= \overbrace{\frac{R^2}{2\pi} \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\psi_0} \frac{\mu}{\ell} \sin\psi d\psi d\alpha}^I + \frac{R^2}{2\pi} \int_{\alpha=0}^{2\pi} \int_{\psi_0}^{\pi} \frac{\mu}{\ell} \sin\psi d\psi d\alpha \\
 &= I + \frac{R^2}{2\pi} \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} \bar{K}(\psi) \mu \sin\psi d\psi d\alpha \\
 &= I + \frac{R^2}{4\pi} \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} \sum_{n=0}^{\infty} (2n+1) k_n \mu P_n(\cos\psi) \sin\psi d\psi d\alpha \\
 T &= \frac{R^2}{2\pi} \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\psi_0} \frac{\mu}{\ell} \sin\psi d\psi d\alpha + R^2 \sum_{n=0}^{\infty} k_n(\psi_0) \mu_n(\theta, \lambda)
 \end{aligned}$$

with (θ, λ) as the spherical coordinates. Recursion formulas for evaluating k_n are found by modifying the formulas for δg because the kernel $[\sin(\psi/2)]^{-1}$ is inherent in the analogous computations for coefficients of δg . The advantage of this expansion in comparison to analogous expansions of Δg and Δk are illustrated in figures 7 and 8 for k_n ($n=2,3,4,5$). These calculations were made by B. Stock on the IBM 370/168 computer.

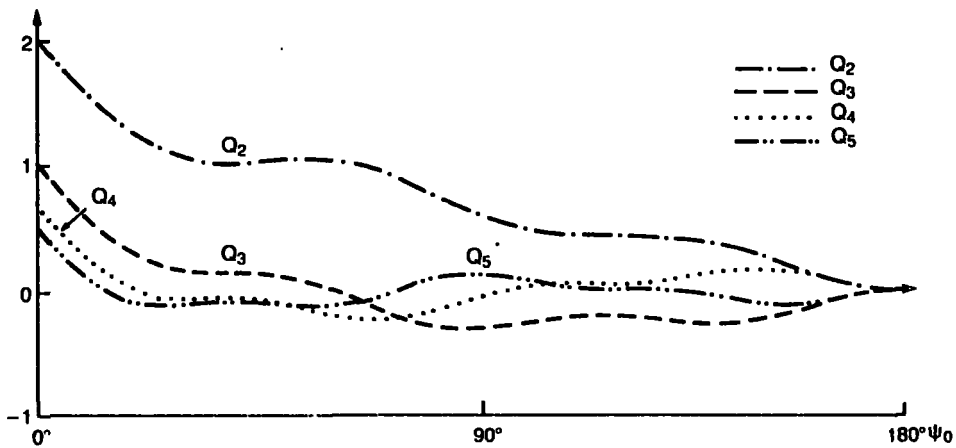


Figure 7.-- Q_n for $n = 2, 3, 4, 5$.

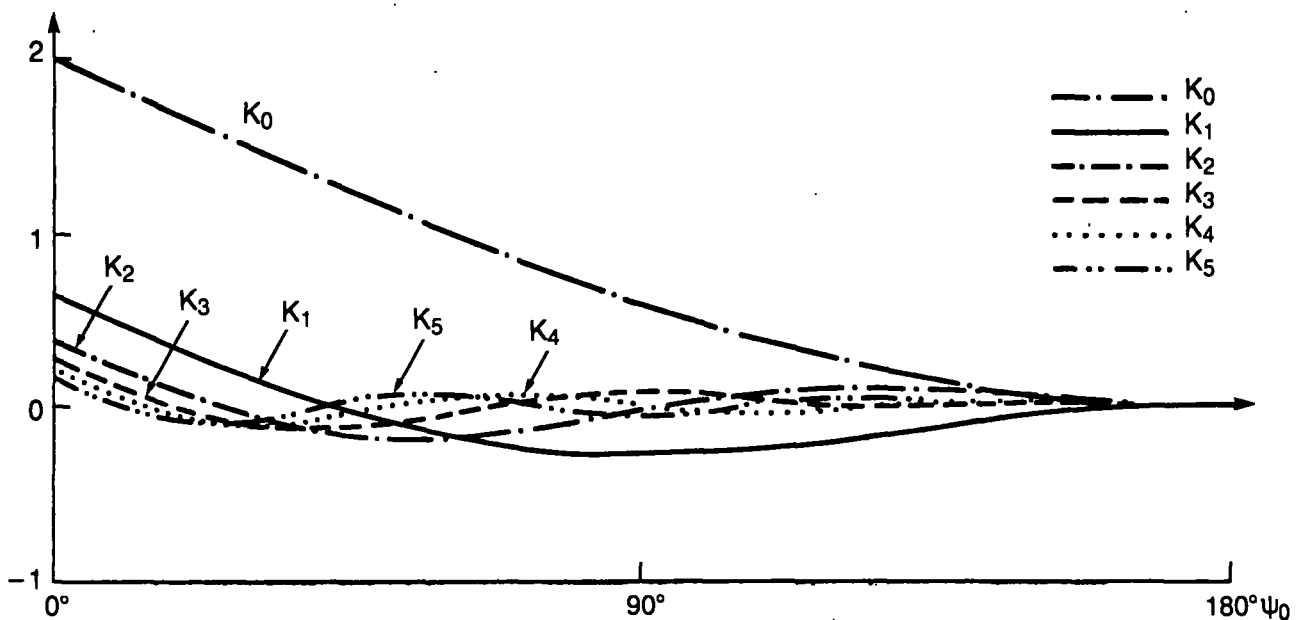


Figure 8.-- k_n for $n = 0, 1, 2, 3, 4, 5$.

The following arguments are also based on his calculations: With $\psi_0 = 1^\circ$ and $\psi_0 = 5^\circ$ we show $Q_n^2 \cdot \sigma(\mu)$ for the GEM 10 model. This is of interest when using formulas such as eq. (31). It is seen that the decrease of the product with increasing n is such that for small values of ψ_0 a remarkable number of terms have to be included in formulas such as (31) to achieve useful results. (See fig. 8.)

To obtain an idea of the orders of magnitudes we computed $K_n^2 \sigma_n(\mu)$ using the GEM model for $\psi_0 = 1^\circ$ and $\psi_0 = 5^\circ$. This is of interest in dealing with formulas such as eq. (31). Table 9 shows that with decreasing ψ_0 the values of $K_n^2(\psi_0) \sigma_n(\mu)$ decrease slowly. On the other hand, for $\psi_0 = 5^\circ$ few terms of the series are of interest for practical application. Figure 9 shows the ratio of K_n/Q_n for $\psi_0 = 1^\circ, 5^\circ$, which is plotted in a smoothed form to illustrate the general trend for $3 \leq n \leq 49$.

Table 9.-- $K_n^2(\psi_0)\sigma_n(\mu)$ values for $\psi_0=1^\circ, 5^\circ$ using the GEM 10 data

n	$K_n^2(\psi_0) \cdot \sigma_n(\mu)$	
	$\psi_0=5^\circ$	$\psi_0=1^\circ$
3	3.981	7.224
4	0.796	1.803
5	0.351	1.026
6	0.148	0.579
7	0.069	0.383
8	0.019	0.161
9	0.008	0.115
10	0.002	0.076
11	<0.0003	0.040
12	"	0.017
13	"	0.023
14	"	0.011
15	"	0.008
16	"	0.005
17	"	0.004
18	"	0.004
19	"	0.002
20	"	0.002
21	"	0.002
22	"	0.001
23	"	0.0003
24	"	<0.0003

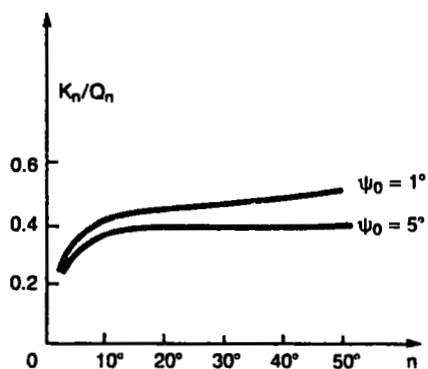


Figure 9.--Smoothed values of K_n/Q_n .

APPENDIX C.--A REMARK ABOUT SECULAR LOVE NUMBERS

Contrary to the secular Love number used by me to describe the global response of the cause of stationary tidal attraction, Munk and MacDonald (1960) introduced a secular Love number describing the response of the yielding cause to centrifugal force. In principal, the latter can be blamed even though it is no longer important.

McClure (1973) gives the following relations for the disturbances of the inertia tensor C_{ij} of the Earth corresponding to the rotation components u_I relative to the rotation axis I

$$C_{13} = \frac{k}{k_s} (A_3 - A_1) u_{1I}$$

$$C_{23} = \frac{k}{k_s} (A_3 - A_1) u_{2I}$$

with A_1 = maximum moment of inertia of the Earth,

A_3 = minimum moment of inertia of the Earth,

k = tidal effective Love number,

k_s = secular Love number,

$$\vec{\omega} = (\omega_1, \omega_2, \omega_3) = \Omega(u_{1I}, u_{2I}, 1 + u_{3I})$$

and Ω being the mean angular velocity of the Earth's rotation referred to a body-fixed terrestrial system. Introducing

$$k_s = \frac{3G(A_3 - A_1)}{a^5 \Omega^2}$$

and the zonal second degree harmonic of the geopotential

$$J_2 = \frac{A_3 - \frac{A_1 + A_2}{2}}{Ma^2} \doteq \frac{A_3 - A_1}{Ma^2}$$

we find

$$k_s = \frac{3GMJ_2}{a^3\Omega^2}.$$

The perturbations c_{ij} in the tensor C_{ij} read

$$C_{ij} = \begin{pmatrix} A_1+c_{11} & c_{12} & c_{13} \\ c_{12} & A_2+c_{22} & c_{23} \\ c_{13} & c_{23} & A_3+c_{33} \end{pmatrix}$$

M is the mass of the Earth; a is the mean Earth radius. Assuming that $A_2 = A_1$ we obtain an estimate for k_s . However, if $A_3 - A_1$ is explained by density anomalies in the upper mantle of the Earth the relation

$$(A_3 - A_1)/\mu$$

no longer gives a hint on the yielding of the Earth in response to centrifugal forces caused by Ω . To some extent, numerical values such as (Capitaine 1979)⁴

$$k_s \doteq 0.95$$

are doubtful. Moreover, the corresponding second secular Love number h_0 cannot be determined without hypothesis. Assuming a specific type of body we can, of course, determine h_s corresponding to such a specific model. However, the introduction of hypotheses should be avoided. We cannot sufficiently specify the Earth. h_s is necessary for the reduction of terrestrial data (even the purely geometrical data) if the permanent tidal deformation has to be eliminated. A crude value is $h_0 \doteq 1.96$. A alternative formula within the same framework is given by Leick (1978)

$$\frac{u'}{u} = 1 - k/h$$

⁴Refer to International Astronomical Union Symposium No. 82, Time and the Earth's Rotation, edited by McCarthy and Pilkington (1979) for a discussion of nutation in space and the diurnal nutation for an elastic Earth. McClure (1973) used the old values of Munk and MacDonald (1960). Lambeck (1980) who used recent data obtained $k = 0.942$.

where u' is the deviation of the angular momentum axis \vec{H} from the aforementioned body-fixed reference axis for an elastic Earth; u denotes the analogous deviation from a rigid Earth. Both deviations of polar-motion type refer to forced nutation, h and k , and have the same meaning as previously. The same limitations mentioned before are connected with the application of this formula to a determination of k . However, \vec{H} cannot be observed directly.

The problem associated with numerical values for k has been discussed in various textbooks (e.g., Stacey 1977). The secular Love number used here by me is not quite the same constant as the one introduced by Munk and MacDonald (1960) and used by McClure (1973) and others. Besides these authors, various numerical values were derived for specific density models by Takeuchi, Pekeris, and Accad, as well as others, who yielded frequency-dependent Love numbers (in some cases) such as

$$h_{\infty} = 0.69 \quad k_{\infty} = 0.35 \quad l_{\infty} = 0.11$$

or

$$h_{\infty} = 0.61 \quad k_{\infty} = 0.30 \quad l_{\infty} = 0.08$$

(For a detailed summary, see Melchior (1978, pp. 105-115).)

These elastic Love numbers are functions of the oscillations of the outer core; consequently, the core dynamics causes slight variations in the static (elastic) values. The aforementioned values refer to infinite periods of core oscillation which, however, do not exist. They are not related to the permanent tide problem although they are sometimes used in connection with this problem.

When we compare the various experimented determinations of k using the variations of the Earth's speed of rotation varying between (Melchior 1978, p. 416)

$$0.300 \text{ and } 0.343$$

in the case of the fortnightly tide, and between

$$0.265 \text{ and } 0.301$$

for the monthly lunar tide, then it is realized that it is better to avoid any reductions of high-precision geodetic data involving such parameters with uncertainties of more than 10 percent. In the case of ℓ the situation is still worse. We know too little about the zero frequency behavior of the Earth. We end up with the final conclusion: assuming, besides hydrostatic equilibrium, that the viscous and elastic responses of the Earth do not depend on the amplitude of the deformation, the use of fluid Love numbers

$$h_f = 1.934 \text{ and } k_f = 0.934$$

as proposed by Lambeck (1980, pp. 26-29) is justified. Otherwise, the values

$$0.28 \leq k \leq 0.934$$

$$0.6 \leq k \leq 1.934$$

seem to be appropriate. The permanent tide must be taken into account fully whenever dynamic interpretations of the flattening are of interest. (See, e.g., Lambeck (1980, p. 26).) To prevent inconsistencies, any corrections in geodetic applications by which the flattening of the Earth is modified should be avoided.

Although the centrifugal force behaves analogously to the permanent tidal force, the amplitudes of both forces are so different that the fluid Love numbers derived from centrifugal forces are not necessarily useful parameters in permanent tide investigations.

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